THE THINGS I SHOULD HAVE DONE, I DID NOT DO*

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It is said to have been an edict of Pythagoras, never to let sleep descend until the events of the day had been thrice reviewed. In a humble way I like to think myself a follower of Pythagoras. The edict therefore seemed to suggest topics that I might discuss on this occasion, although, I must say, the analogy between sleep and my retirement from the presidency of the Association does not strike me as particularly apropos.

Projects in which the Association might interest itself are surely appropriate to our concern. If it seems incongruous for me to bring such projects forward at the instant when I have safely stepped from the position that might well seem to have been peculiarly advantageous for action, I must concede the point. In extenuation I shall let Samuel Johnson (The Rambler, April 14, 1750) speak for me. He says: "If the most industrious of men could recall his past moments, it is scarcely to be imagined how few would be marked out by any permanent effect; how small a proportion his action would bear to his possibilities; how many chasms of vacuity he would find even in the most tumultuous hurries of business and the most eager vehemence of pursuit. Modern philosophers say that the hardest of bodies are so porous, that, if all matter were compressed to perfect solidity it might be contained in a cube of a few feet. In like manner, if all the employment of life were crowded into the time which it really occupied, perhaps a few weeks, or days, or hours, would be sufficient for its accomplishment. For such is the inequality of our corporeal to our intellectual faculties, that we contrive in minutes what we execute in years, and the soul often stands an idle spectator of the labour of the hands and the expedition of the feet."

A few years ago there was displayed at the Art Institute of Chicago a painting by a great American artist (Ivan L. Albright). This painting, though very simple, has become widely renowned. It pictured merely the door of a tomb with a wreath that had been deposited by a withdrawing hand. The title of the picture was "The things I should have done, I did not do." You see that I am acknowledging my plagiarism of the title of this address.

Our Association was conceived by its founders as an agency to operate in the collegiate field, and over the years it has continued to profess the concern for matters in this field to be its especial object. This assures me my franchise for what I wish to discuss, and leaves me only to hope that you will find it worth while. That it is original I will not claim. However, in changing times even the raising of oft-discussed questions need not be unprofitable. For under new realignments of prospects and purposes these may well be invested with enhanced importance.

The topics I have chosen are three; first, our elementary college mathematics curriculum under the impact of the widespread trend toward more general

* Presented as a presidential retiring address before the Mathematical Association of America at Minneapolis, September 3, 1951.

education at the expense of specialized subjects; second, our concerns and practices in the training of secondary school teachers; and third, our response to the unprecedented expansion of employment opportunity for mathematically trained personnel.

The great majority of college students are restricted in their choice of mathematics courses to such as we may, perhaps with reservations, refer to as traditional. They are courses in which the essentials of algebra, trigonometry, the calculus, *etc.* are more or less conventionally set forth, in accompaniment with drill that is directed toward technical proficiency. We hear the charge now and then, that these courses are set and dry, that they are withdrawn from matters of immediate vital interest, and that they are centered upon narrow technological objectives, to the extent that they are unrewarding except for a few. In large measure such criticism is certainly mistaken. In the main, these courses are in continual revision. One can claim for them much richness, both in content and method, that is contemporarily wholly to the point. The needs professed by innumerable civilian agencies, and no less by the nation's armed forces, as was emphasized by the recent countrywide examinations, must certainly be considered as eloquent of the profit with which a student may spend time in mathematics courses.

Let us not, however, be smug about this. There is, perhaps, some justice in the criticism of our courses, especially that they are designed for students directed toward technical fields. Can we honestly say that they are not wide of the optimum for students otherwise directed, and especially so for those who can give only a year to mathematical study? The importance of this seems to me to be rising. Over the country there has been, and is, a decisive trend in the colleges toward what is called "more general education." There has been a pretty wide-spread growth of sentiment favoring a broader and less specialized basis of collegiate preparation, in recognition of students' predestined roles as citizens and educated people. There is concern for humanistic and especially for social subjects. Many schools have instituted heavily prescribed courses in so-called "integrated liberal studies."

Now in such courses mathematics invariably finds no place at all, or only a vanishingly insignificant one. Indeed in some places such courses have operated to rule out mathematical study. I will not labor the anomaly of this, at a time like the present, when war and revolutionary scientific and technological advances have made mathematical intelligence an ascendant agent in the molding of our very way of life, and an essential to its preservation. It is, however, the fact, that while newer means of communication are bringing more and more people into daily contact with the quantitative aspects of civilized life, the colleges are pouring forth great majorities of graduates who have only vaguely erroneous or non-existent notions of the nature and significance of our subject. This being so, is it surprising that in the secondary schools mathematics as a curricular subject is, to say the least, on the defensive?

I know, of course, that this problem is not new, and that serious and able

attempts to confront it have been made. If we admit, as I think we must, that our elementary college courses are patterned for technical mastery, must we not also admit that only by an extraordinary coincidence could they at the same time be best adapted for students with liberal objectives? This has engendered a great deal of conviction that something else is needed. In response to it there has been a wealth of publication, much of it distinctly admirable, of professedly non-technical books on the meaning of mathematics, on mathematical surveys, and the like.

I fear the definitive answer to the challenge has not yet been found. I am at least unaware of any general acceptance, either by us or by our critics, of any response that has yet been proposed. We have tried integrations of subject matter—and many regard them as little more than mixtures yielding but little to the purpose. We have tried surveys—and many have found them uninforming and superficial. We have tried new and unorthodox subject matter without discovering many new virtues, and we have tried an emphasis upon an appreciation of mathematics, only to find that appreciation can hardly be called forth by what is unknown and unmastered.

Much time and enthusiasm have gone into these experiments. The returns have generally not been commensurate. All too often the experimental course has been handicapped from the start by its attraction for those students who hoped to find in it the easy course in mathematics.

Has the problem, then, no solution? We must not think that, nor must we cease to apply ourselves to it. College students by thousands are constantly choosing their electives. We cannot be content to have them leave mathematics aside. Rich in its past, dynamic in the present, prodigious for the future, replete with simple and yet profound ideas and methods, surely mathematics can give something to anyone's culture. Does it seem, on the face of it, to be impossible to broach some of this even in a single year's terminal course?

I am sure that the key to the way of doing this is not a golden one that some lucky seeker will accidentally find. It will require a step by step search, and the approach to it will probably be hastened by combined effort. There are, over the country, many small colleges, junior colleges, and teachers colleges, in many of which this problem is more pressing than it is in the universities. It is not a problem for local solution. Therein, it seems to me, lies an opportunity for the Association. To find and enlist the talent that will apply itself, to organize it, to finance it, and in the end to publish its findings, therein may be a chance for great service. Can we, until this has been done, say that we have given the problem the best we are capable of?

I turn to my second topic: our concern with and practice in the training of secondary-school teachers. What a lack of uniformity prevails there! Over the country mathematics teachers are being turned out by institutions ranging from two-year colleges to the universities, and with trainings that may have terminated with the end of the first year of the calculus or with the Ph.D. degree. With such divergence of practice, what standards can there be? By what yardstick

shall an administrator appraise his staff? This is an important matter. It is, however, not the one to which I wish now to speak.

It is my own experience that, on the whole, we treat our teacher training responsibilities pretty casually. The plan of our courses, in college and graduate school, is by and large that of a road which penetrates deeper and deeper into the mathematical domain and eventually leads, through one branch or another, to the frontier. Subconsciously perhaps, but none the less truly, we think of ourselves first as trainers of mathematicians, and content ourselves all too easily with turning out as teachers those we judge to lack the qualifications for the higher estate.

If this is so, can we wonder that our high-school teachers widely fall short in mathematical mastery, even in the branches which concern them most, and that they are themselves so conscious of this that they regard this Association as too "high-brow" for them and consider the MONTHLY as "over their heads." One hears that this is not so in Europe. But we see our mathematics teachers content themselves with their shortcomings. They attend university summer schools in numbers, but to take degrees in Education, not to ameliorate their mathematical ineptnesses. It must be said our teachers are rarely at ease with their subject. Yet they are the ones upon whom we must depend to broach and initially interpret mathematics to the ever on-coming stream of the newer generation.

I believe we are at fault for this condition, and that it roots appreciably in our cavalier attitude toward the job of teacher training. Teaching, at whatever the level, is a dignified profession. And it calls at each level for especial qualifications. Many of our strongest mathematicians would cut but sorry figures before a high-school class. Is it more damning than this that many young men and women who are talented for teaching are less so for the abstractions of higher mathematics?

Do we recognize this in our practice? I think insufficiently so. In the main we send our destined teachers unselectively through mathematicians' courses, and trouble ourselves hardly at all over whether these are so specialized as to offer little in the way of perspective, or anything for future effective teaching. We stand guard that these students pass in partial differentiation and multiple integration, while we concern ourselves little with their arithmetical misconceptions, and condone the meagerness of their geometric mastery or skill. We permit many of them to leave us burdened with a conviction that mathematics is a painful and exacting discipline, a conglomerate of many sparsely related subjects, of which the most are soon and well forgotten. We give them almost no appreciation of the subject's unity, of the sublimity of even its elementary branches, or of the fact that it exists in response to an ineradicably ingrained human need and quest. We show them the trees, and disregard that they never see the forest.

These things have, of course, been said before. It is even a familiar cry that training for secondary-school teaching should feature breadth and comprehension rather than depth and specialization. The MONTHLY contains excellent

papers on this theme. A "Report on the Training of Teachers of Mathematics" by a commission of the Association (The MONTHLY, May, 1935) made an eloquent plea for differentiated teacher training extending even to the Ph.D. degree.

I do not mean to say that no special courses for teachers or differentiation of practice presently exist. The point I wish to make is mainly this: that there is need for more than just local attention to this problem. Therein may be an opportunity for the Association. Divided effort lacks authority and is unequal to the scope of the problem. With coordinated and sustained attention, on the other hand, we may perhaps in time establish norms of good practice which the schools could not afford to ignore, and through which our teachers might develop an *esprit de corps* for mathematics rather than for method.

I come to my third and final topic. During the last decade we have seen a great widening of the field in which mathematically trained persons are sought for. Our graduates at all levels therefore have more diversified opportunities for employment—even for careers. Where, not long ago, students of advanced mathematics had only small choice but to enter the teaching profession, they have now manifold opportunities in utilities and industry, and in the armed and civilian services of the Government. Indeed, this change has been so marked that the teaching profession is threatened with soon finding itself in straights to recruit enough of the best young mathematicians to insure the perpetuation of essential training staffs for the future.

What I have in mind in connection with this is: Granted the change, what responses are we, as teachers, making to it? Diversity of opportunity entrains a diversity of objectives. These, it might be expected, will call for diversified training, unless we are complacently to resign our students to the necessity of sharply reorienting themselves upon leaving us, and of hastening to learn many things we did not teach them. I do not know how widely these matters are calling forth curricular revisions. At the University of Wisconsin we have a program of relatively recent origin which seems pertinent to this. Since I have some knowledge of this I should like to say a little about it.

The usual graduate of an engineering school has neither the mathematical training nor the breadth of scientific experience to qualify him for more than routine engineering practice. He is not equipped to be of much use in what we may style engineering research. The usual mathematics student, on the other hand, is quite ignorant of both the theory and the practice of engineering pursuits. Now much of the enlarged employment opportunity lies precisely in the domain of the applications of mathematics in engineering. We at Wisconsin, therefore, have established a four or five year course, which is administered by a committee of mathematicians and engineers, and which leads to the bachelor's or the master's degree in "Applied Mathematics and Mechanics." The course is largely prescribed, and enrollment in it is restricted to students who maintain a distinctly superior average in their mathematics courses.

The freshman and sophomore years of the course are not especially distinc-

tive. They include a year of the calculus, and along with year courses in physics, chemistry and engineering mechanics, the usual courses in English, social studies, *etc.* In the junior year the technical training is stepped up. This includes courses in advanced calculus, light, electric circuits and electro-magnetic waves, along with engineering courses in the mechanics of materials, electrical machinery, and hydraulics. The courses of the senior year cover mathematical mechanics, vector analysis, Fourier analysis and some complex variable, along with aerodynamics, thermodynamics, and mechanical vibrations. The fifth year, if it is taken, gives considerable opportunity for concentration in one or another engineering field.

My purpose in giving this outline has been mainly to show that teaching adjustments are being made. Probably many others have been made, and are being made, in many places. Now I believe that a great service to mathematical education might be rendered by bringing this whole matter under competent and continued study, by comparing and collating the various adjustments and by publicizing them. Successful innovations could thus be brought to the attention of all. May there not be an opportunity for the Association along this line?

I bring my remarks to a close. In making them I have had no intention to be destructively critical, but only to bring under focus some issues which seem to me to be worth while. The question we must pose ourselves in connection with them is, I believe, as Hamlet has put it:

"Whether 'tis nobler in the mind to suffer
"The slings and arrows of outrageous fortune,
"Or to take arms against a sea of troubles,
"And by opposing end them."

AN EQUATION IN GAUSSIAN INTEGERS

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This paper is concerned with the equation $ax^2+by^2+cz^2=0$ which was investigated by Legendre when the coefficients are rational integers. He reduced the equation to a "normal form" in which

(i) a, b, c are square-free,

and

(ii) a, b, c are co-prime in pairs.

He found that necessary and sufficient conditions in this case for the equation to be soluble, *i.e.* that a solution in integers x, y, z, not all zero, exists, were that

(a) a, b, c are not all of the same sign,

(b) $-bc, -ca, -ab$ are quadratic residues of a, b, c respectively.

H. J. S. Smith [1] noticed that these conditions carry over to the case when the coefficients and unknowns are Gaussian integers, *i.e.* of the form $U+iV$ where U and V are rational integers. We show now that the proof given in Dickson's book [2] of Legendre's result can be extended to this case.

The equation may be taken to be in the normal form of (i) and (ii). First, we may take a, b, c to be square-free, for if $a=A\alpha^2$, $b=B\beta^2$, $c=C\gamma^2$, where A, B, C are square-free, and the equation $A\xi^2+B\eta^2+C\zeta^2=0$ has a solution (ξ, η, ζ), then $(\beta\gamma\xi, \gamma\alpha\eta, \alpha\beta\zeta)$ is a solution of $ax^2+by^2+cz^2=0$. Secondly, a, b, c can be taken to be prime in pairs. We can take $(a, b, c)=1$ and assume that a and b have a (Gaussian) prime factor p in common; then $cz^2\equiv 0 \pmod{p}$, which gives $z\equiv 0 \pmod{p}$ and therefore $z=pZ$, and our equation becomes $(a/p)x^2+(b/p)y^2+pcZ^2=0$, an equation where the product of coefficients is abc/p and so has a smaller norm than abc, and no two of the coefficients are divisible by p. After a finite number of steps we have coefficients that are co-prime in pairs.

Our result is the following:

THEOREM. *A necessary and sufficient condition for the equation*

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$

to be soluble in Gaussian integers, not all zero, if (i) *and* (ii) *are satisfied, is*

(iii) *bc, ca, ab are quadratic residues of a, b and c respectively.*

Note. We can dispense with the minus sign of Legendre's condition because $i^2=-1$.

Suppose the equation has a solution x, y, z. We may assume that x, y, z have no common factor and this implies that x, y, z are co-prime in pairs. For if $p \mid x, p \mid y$, where p is a Gaussian prime, $cz^2\equiv 0 \pmod{p^2}$, $z^2\equiv 0 \pmod{p}$ and then $p \mid z$ also. Then $ax^2+by^2\equiv 0 \pmod{c}$, which gives ab as a quadratic residue of c, because $(x, c)=(y, c)=1$. The condition is thus necessary.

To prove its sufficiency, we need a preliminary result.

LEMMA. *Conditions* (i) *and* (ii) *exclude the case* $|a|=|b|=|c|>1$.

For suppose $|a|=|b|=|c|>1$; then $N(a)=N(b)=N(c)$ where $N(a)$, the norm of a, is a rational integer. Firstly $N(a)$ cannot be divisible by a prime $p\equiv 3 \pmod{4}$ since then $p \mid a, p \mid b, p \mid c$ which is false. If $N(a)$ is even, a, b, c have a common factor $1+i$, which is again false. If $N(a)$ has a prime factor $p\equiv 1 \pmod{4}$, then a contains a factor π or $\bar{\pi}$, where π is a Gaussian prime such that $\pi\bar{\pi}=p$. At least one of the pairs a, b; b, c; c, a has either π or $\bar{\pi}$ as a common factor, again contradicting (ii).

We see easily that the case $|a| = |b| = |c| = 1$ always gives us a soluble equation.

By symmetry, we may take $|a| \leqq |b| \leqq |c|$, where now at least one of the equality signs can be omitted. Then $|ab| \leqq |ac| \leqq |bc|$.

We define $|ac| = J$ to be the *index* of equation (1). If we do not make the assumption that $|a| \leqq |b| \leqq |c|$, then the index is the middle one of $|ab|$, $|ac|$, $|bc|$ if they are all different, or the common value of two of them otherwise. The two definitions are equivalent.

The proof will be by induction on J. The values of J^2 form a sequence of positive integers, $J_1, J_2, \cdots, J_n, \cdots$ and we show that if the result holds for $J^2 \leqq J_n$, then it holds for $J^2 \leqq J_{n+1}$.

The case $J = 1$ has been considered. Now assume $J > 1$ and that equation (1) can be solved for all possible values of the index $< J$. By condition (iii), there exists an integer R such that $R^2 \equiv ab \pmod{c}$. By condition (ii), there exists an integer r such that $ar \equiv R \pmod{c}$. We can take r such that $0 < |r| < (1/\sqrt{2})|c|$.

Clearly $ar^2 \equiv b \pmod{c}$ and so

$$ar^2 - b = cQ \tag{2}$$

where Q is an integer and

$$|Q| = \frac{|ar^2 - b|}{|c|} \leqq \frac{|ar^2|}{|c|} + \frac{|b|}{|c|} \leqq \tfrac{1}{2}|ac| + 1 = \tfrac{1}{2}J + 1 < J$$

if $J > 2$.

We must now consider the cases $1 < J \leqq 2$, *i.e.* $J = \sqrt{2}$ and $J = 2$.

If $J = \sqrt{2}$, equation (1) always has a solution. We have $a = \epsilon_1$, $b = \epsilon_2$ or $\epsilon_2(1+i)$, $c = \epsilon_3(1+i)$ where ϵ_1, ϵ_2 and ϵ_3 are units. Further, $b = \epsilon_2(1+i)$ is impossible since b and c are co-prime. In the other case the equation always has a solution, as can be seen by enumerating cases.

If $J = 2$, $|ac| = 2$. Either $|a| = |c| = \sqrt{2}$, whence $|b| = \sqrt{2}$, an impossible case, or $|a| = 1$, $|c| = 2$ and thus c has a factor $(1+i)^2$ and so is not square-free, contradicting (i).

We now consider $J > 2$.

If $Q = 0$, $ar^2 = b$ and $|r| = 1$ since b is square-free. This gives $a = \pm b$ and then we always have a solution of (1).

If $Q \neq 0$, let A be the greatest common divisor of the three terms in (2). Since $A \mid b$, we have $(A, a) = (A, c) = 1$; thus $A \mid Q$ and $A \mid r^2$. Hence $A \mid r$, since A divides b and so is square-free. Put $r = A\alpha$, $b = A\beta$, $Q = AC\gamma^2$, where C is square-free. Substitution in (2) now gives

$$aA\alpha^2 - \beta = cC\gamma^2 \tag{3}$$

where the three terms are relatively prime in pairs. We also put $B = a\beta$.

We shall presently consider the equation

$$AX^2 + BY^2 + CZ^2 = 0 \tag{4}$$

and show now that A, B, C satisfy conditions (i), (ii) and (iii) and that equation (4) has a smaller index than (1). Finally we will show that the solution of (4) leads us to a solution of (1).

None of A, B, C is zero. From $AB = Aa\beta = ab$ we see that A and B are relatively prime and also square-free. From (3), C is prime to $\beta \cdot aA = AB$. Again from (3),

$$cC\beta \text{ is a quadratic residue of } aA,$$

$$aAcC \text{ is a quadratic residue of } \beta,$$

$$AB \text{ is a quadratic residue of } C.$$

Now ac is a quadratic residue of $b = A\beta$, and so of A. Thus $cC\beta$ and ac are quadratic residues of A and so is their product, βaCc^2. Hence $\beta aC = BC$ is a quadratic residue of A.

Since ac and $acAC$ are quadratic residues of β, AC is a quadratic residue of β. Since βcC and βAc are quadratic residues of a, so is $(\beta c)^2 AC$. Further, a and β are relatively prime, and so AC is a quadratic residue of $a\beta = B$.

A, B and C thus satisfy the same conditions as a, b and c.

We next show that the index of (4) is smaller than J if $|b| < |c|$. We have

$$|AC| \leqq |AC\gamma^2| = |Q| < J,$$
$$|AB| = |ab| < J,$$

and the index of (4) is $< J$, by the alternative definition of index.

In case $|b| = |c|$, so that $|AB| = |ab| = J$, we distinguish two cases:

(a) $|BC| < J$. The index of (4) is $< J$.

(b) $|BC| \geqq J$. We have $|AC| < |AB| \leqq |BC|$, and so $|A| \leqq |C| < |B|$. In this case we apply the whole reduction process to the equation $AX^2 + CY^2 + BZ^2 = 0$ (where we note the interchange of B and C) to get $A'X^2 + C'Y^2 + B'Z^2 = 0$ whose index is smaller than J since $|C| < |B|$, analogous to $|b| < |c|$.

If the index of equation (4) is $< J$, a solution X, Y, Z exists by hypothesis. Putting

$$\begin{cases} x = A\alpha X - i\beta Y \\ y = iX + a\alpha Y \\ z = C\gamma Z \end{cases} \tag{5}$$

and using (3) and the definition of B, we have

$$ax^2 + by^2 + cz^2 = cC\gamma^2(AX^2 + BY^2 + CZ^2) = 0. \tag{6}$$

If the index of (4) is $= J$, we find a solution of (4) from that of $A'X^2 + C'Y^2 + B'Z^2 = 0$ by the above substitution, and a second substitution then gives a solution of (1).

Further, the solution (5) of (1) is non-trivial if X, Y and Z are not all zero. For if in (5) $x=y=0$, elimination of X gives $(aA\alpha^2-\beta)Y=0$, where the first factor is $\neq 0$ by (3). Thus $Y=0$ and therefore $X=0$, giving $Z=0$ also, which is false.

Equation (1) thus has a non-trivial integer solution if conditions (i), (ii), (iii), are satisfied. This completes the proof.

I have tried in vain to find real positive numbers l, m, n depending on a, b, c such that a solution of (1) exists with $|x| \leqq l$, $|y| \leqq m$, $|z| \leqq n$, say a minimal solution. It does not seem easy to relate the minimal solution of $AX^2+BY^2+CZ^2=0$ to the minimal solution of $ax^2+by^2+cz^2=0$. Holzer [3] and Mordell [4] have recently studied minimal solutions of $ax^2+by^2-cz^2=0$ with rational integer coefficients, where a, b, c are positive. Holzer found that a solution exists with $|x| \leqq \sqrt{bc}$, $|y| \leqq \sqrt{ca}$, $|z| \leqq \sqrt{ab}$. Mordell found a similar, but slightly weaker, result by a very much simpler method than Holzer's. Neither of their methods seems to apply to the complex case.

Instead of the Gaussian field, we may consider the field $k(\sqrt{-d})$, where d is a positive square-free integer and the field has a Euclidean algorithm. We find on determining R and r that $|r| \leqq \frac{1}{2}\sqrt{d+1}\,|c|$ if $-d \equiv 2, 3 \pmod 4$. This gives $|Q| \leqq \frac{1}{4}(d+1)J+1$ and then this can be $<J$ only if $d=2$. The argument given above will then work if $J>4$, leaving the cases $1<J\leqq 4$ to be considered separately—a tedious though probably not difficult matter. If $-d \equiv 1 \pmod 4$ we find that $|r| \leqq \frac{1}{4}\sqrt{d+4}\,|c|$, $|Q| \leqq \frac{1}{16}(d+4)J+1$. This can be $<J$ for $d=3, 7, 11$, and once more the argument will work if J is sufficiently large, leaving various special cases to be considered separately. The minus sign of Legendre's original conditions would have to be reinstated also, since the equation $x^2=-1$ is no longer soluble.

I wish to thank Professor L. J. Mordell for his helpful criticism and advice in the preparation of this paper and also the Department of Scientific and Industrial Research for a Maintenance Allowance.

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FUNCTIONS OF N VARIABLES IN EXTENDED ANALYTIC GEOMETRY

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Foreword. The general subject here called "extended analytic geometry" is concerned with the effort to obtain, by means of a system of axes through a common origin, actual "drawable" loci of equations in n variables even when n exceeds three. It seemed evident from the beginning that the results, if sufficiently simple, should be helpful in at least such phases of mathematics as the treatment of simultaneous equations. While the possibilities in that field still seem the most promising, other uses have appeared.

Now, in the interval following acceptance of this paper the writer has found what appears to be (so simple and obvious it is in the light of "hindsight") the "best possible" spacing of axes for the purposes discussed above. While it is not the system for which the figures of this paper have been drawn, it may be said that fortunately the techniques developed in this and the foregoing related papers are essentially independent of the angles between the axes. The latter merely determine the particular equations of transformation for the case considered, and affect the basic methods only in the complication or simplification of the algebra. Since the publication date is drawing near it has seemed best to retain in its original form the part of the article dealing with functions of three variables (Sections 1–6). In any case the solids and surfaces obtained when the axes on the plane are 60° apart are much like those found for orthogonal axes, and the full importance of the change to the latter system does not appear until the number of axes on the plane exceeds three.

1. Introduction. A function U of the variables $x_1, x_2, \cdots, x_n$ has of course a simple geometric interpretation, via plane and solid analytic geometry, when $n=1$ or 2. But when $n>2$ the situation in orthodox mathematics has changed drastically. Geometry has absconded or gone abstract, as one wills, and only algebra remains. To the writer this seems especially unfortunate in connection with the interpretation of maxima and minima of functions of several variables. It is the purpose of this paper to suggest a geometric background for the treatment of such problems, with special reference to functions of three and four variables.

2. The three-dimensional system. In plane analytic geometry as extended by the writer's method,* the axes x_1 to x_n lie on a common plane and pass through a common origin. In the solid variant (Paper I, p. 261) an additional axis, called here the U-axis, passes through the origin and is perpendicular to the n-axes plane, with its positive direction upward.

By definition the locus of the equation

$$U = f(x_1, x_2, \cdots, x_n) \tag{1}$$

is the totality of points having coordinates $(x_1, x_2, \cdots, x_n, U)$ which satisfy (1).

It develops that the locus of (1), when it exists, is normally a solid such as, for example, a filled cone; but that exceptionally it may be a surface or some other degenerate form. The solid figure may fill all space, as in the case of most linear

* Two preceding papers on the subject, published in this MONTHLY and hereafter referred to as Papers I and II, are:

I. An Analytic Geometry for N Variables, vol. 52, May, 1945.

II. Some Applications of Extended Analytic Geometry, vol. 56, March, 1949.

functions; but when maxima or minima exist the locus usually has a definite top or bottom. Considering it as a family of surfaces, we may find the equation of the bounding surface, in a very general case, in terms of ordinary rectangular coordinates X, Y, and Z. (As in previous papers, X and Y are the coordinates of the general point on the n-axes plane with respect to a superimposed system of two axes.) Thus the algebraic criteria for maxima and minima of a function of n variables, which become more and more complicated as n increases, are replaced by those for a function of two variables. Moreover, they often need not be applied at all when the shape of the bounding surface is known.

For simplicity we shall confine much of this paper to a discussion of the loci of

$$U = f(x, y, z). \tag{2}$$

The procedure for the general case will be illustrated first as applied to the special equation

$$U = x^2 + y^2 + z^2. \tag{3}$$

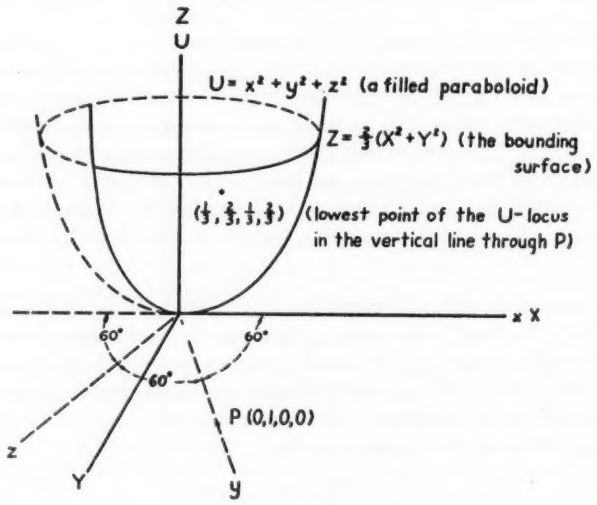

FIG. 1

Consider the point $P(0, 1, 0)$ on the 3-axes plane, which becomes $P(0, 1, 0, 0)$ in the space system (Figure 1). Now the G.C. (generalized coordinates) of P are $(x, 1-x, x)$,§ so that the value of U *over this point* is determined by the equation

§ The G.C. of the point (a, b, c) are $(x, a+b-x, c-a+x)$ by II, (8), p. 160.

$$U = x^2 + (1 - x)^2 + x^2 = 3x^2 - 2x + 1. \tag{4}$$

Evidently the minimum value of U over P is $\frac{2}{3}$, as found by calculus, while there are values of x, and hence coordinates of P, which yield all real values of U exceeding $\frac{2}{3}$.

To get the equation of the bounding surface we may use the general point (X, Y) instead of the specific one selected above. Since, from the fundamental relations [II, (2), p. 159],

$$2X = 2x + y - z, \qquad 2Y = \sqrt{3}(y + z), \tag{5}$$

the G.C. of (X, Y) are

$$(x, Y/\sqrt{3} + X - x, Y/\sqrt{3} - X + x), \tag{6}$$

as obtained by eliminating z and y successively from (5). Substituting in (3) and simplifying, we have

$$U = 3x^2 - 4xX + 2\left(\frac{Y^2}{3} + X^2\right), \tag{7}$$

whence $x = 2X/3$ for a minimal U. When the latter is denoted by Z, we get from (7)

$$Z = \tfrac{2}{3}(X^2 + Y^2). \tag{8}$$

Thus the locus of (3) is the "filled paraboloid" (8).

The step to the general method for a function of three variables is obvious, though of course the way has not been prepared for a rigorous theorem covering all contingencies. In the specific case above the paraboloid (8) is clearly the bottom of the locus, since from (7) $(\partial^2 U/\partial^2 x) > 0$. Of course on occasion the surface corresponding to (8) could be the top of the locus or merely a special surface within it. Usually it is easy to make the decision; but for elusive functions the "posthole method" illustrated above for the point (0, 1, 0) may aid in the pictorial diagnosis. With this understanding we may state the general method.

3. Working rule: If the locus of the equation

$$U = f(x, y, z) \tag{2}$$

has a bounding upper or lower surface, the equation of this surface in terms of rectangular coordinates (X, Y, Z) may normally be found by the procedure hereafter described.

(a) Replace x, y, and z in (2) by t, $K+X-t$, and $K-X+t$ respectively.

(b) Set $\partial U/\partial t = 0$ and solve, thus obtaining t, and hence x, y, and z, in terms of X and K.

(c) Substitute in (2), replacing U by Z and, after simplifying, K by $Y/\sqrt{3}$.

Interesting illustrations are afforded by functions (9) and (11) below.

The locus of

$$U = x^2 + y^2 + z^2 - xy - xz - yz \tag{9}$$

is the filled parabolic cylinder

$$4Z = (Y - X\sqrt{3})^2, \tag{10}$$

which touches the XY plane from above on the line $Y - X\sqrt{3} = 0$ (Figure 2). From (5), the equation of this line in terms of the original variables is $z = x$.

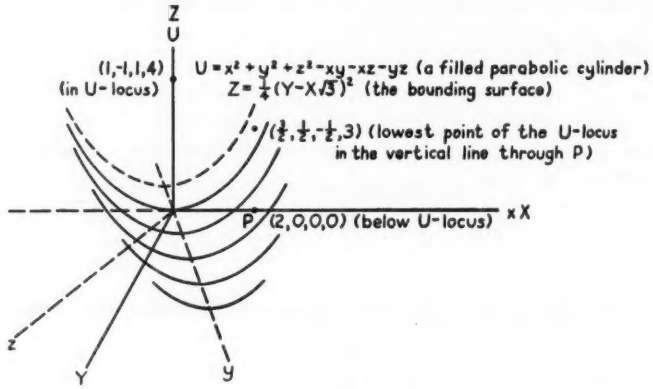

FIG. 2

Thus $U = 0$, its minimum, along this line when $x = z = a$ and $U = (y-a)^2$ with $y = a$; that is, when $x = y = z =$ an arbitrary constant a.

For the equation

$$U = x^2(y + z) \tag{11}$$

the locus is a "double wedge" of which one half fills the region above the XY plane and in front of the XZ plane. Thus half of the plane $Z = 0$ is a minimal surface and half is maximal. In this case the necessary algebraic conditions for extremal values of U yield $x = 0$ and $y + z = 0$; but these equations tell little without the geometric interpretation, since clearly U may have any real value.

4. Loci on the 3-axes plane. An incidental result from the foregoing working rule is a more general method than that previously given for obtaining the locus of

$$f(x, y, z) = 0 \tag{12}$$

on the 3-axes plane. In I, page 257, the equation of the bounding curve about the typical locus area is obtained, when f is a second degree function, by making the substitutions (6) in (12) and then equating to zero the discriminant of the

quadratic in x. But now whenever, with reference to the solid $U=f(x, y, z)$, the equation $Z=F(X, Y)$ of the bounding surface is obtainable by the general rule, we merely set $Z=0$ to find the equation of the curve bounding the locus of (12).

5. Special loci in space. It was stated above that exceptionally the locus of a function of n variables, when $n>2$, may be a surface instead of a solid. Indeed, any equation $Z=F(X, Y)$ of a surface may be changed to the form $Z=f(x_1, \cdots, x_n)$ by substitutions (7), p. 256, in I. We shall now find criteria for the abnormal loci.

DEFINITION: *The locus of*

$$U = f(x, y, z) \tag{2}$$

is degenerate if, for every real and non-isolated point (a, b, c, U_1) *that it contains, the curves* $U_1=f(x, y, c)$, $U_1=f(x, b, z)$, *and* $U_1=f(a, y, z)$ *are identical.*

Note that the locus of (2) will be degenerate if it is non-existent (imaginary) or a surface, curve, or point. Conceivably, though improbably, the conditions of the definition might hold on concentric shells of some highly unusual solid loci.

THEOREM 1. *When*

$$U_y \equiv U_x + U_z \tag{13}$$

the locus of $U=f(x, y, z)$ *is degenerate.*

Proof: Assume that $U_1=f(a, b, c)$, where (a, b, c, U_1) is not an isolated point on the locus. Consider the curve $U_1=f(x, y, c)$ in the plane $Z=U_1$. On this curve it follows from (5) that $dY/dX=(\sqrt{3}dy)/(2dx+dy)$, where dY/dX is the conventional slope of the curve in the plane. The result may be written in the form

$$\left.\frac{dY}{dX}\right]_{z=c} = \left.\frac{\sqrt{3}\,\dfrac{\partial y}{\partial x}}{2+\dfrac{\partial y}{\partial x}}\right]_{z=c}.$$

Similarly,

$$\left.\frac{dY}{dX}\right]_{y=b} = \left.\frac{\sqrt{3}\,\dfrac{\partial z}{\partial x}}{2-\dfrac{\partial z}{\partial x}}\right]_{y=b}.$$

But since $\partial y/\partial x=-(U_x/U_y)$ and $\partial z/\partial x=-(U_x/U_z)$, and since $U_y\equiv U_x+U_z$ by hypothesis, it follows that on the given plane

$$\left.\frac{dY}{dX}\right]_{y=b} = \left.\frac{dY}{dX}\right]_{z=c}$$

wherever these quantities are defined. That is, the curves $U_1=f(x, b, z)$ and $U_1=f(x, y, c)$ are identical and coincide (by a similar argument) with the curve $U_1=f(a, y, z)$.

For an illustration, consider the equation

$$U = x^2 + y^2 + z^2 + xy - xz + yz - a^2. \tag{14}$$

Here (13) holds, so that the locus is degenerate. The first step of the working rule yields $U=X^2+Y^2-a^2$ (t drops out), and hence the locus of (14) is the paraboloid

$$Z = X^2 + Y^2 - a^2. \tag{15}$$

In this case the locus of U coincides with that of Z and reaches its minimum $(-a^2)$ when $X=Y=0$. It does not follow, of course, that necessarily $x=y=z=0$ for a minimum U, since the G.C. of the origin are $(x, -x, x)$ and hence U reaches its minimum when $x=-y=z=t$, with t arbitrary.

The converse of Theorem 1 is false, but a partial converse is true, as follows:

THEOREM 2. *If the locus of $U=f(x, y, z)$ is a surface, then $U_y \equiv U_x+U_z$.*

For then

$$U = f(x, y, z) = F(X, Y) = F\left(\frac{2x + y - z}{2}, \frac{\sqrt{3}\,y + \sqrt{3}\,z}{2}\right),$$

whence $U_x=F_X$, $U_y=F_X(\frac{1}{2})+F_Y(\sqrt{3}/2)$, and $U_z=F_X(-\frac{1}{2})+F_Y(\sqrt{3}/2)$. The theorem follows.

6. Special loci on the 3-axes plane. When $U\equiv 0$, the results of Article 5 apply with slight modification for loci on a plane. Theorems 1 and 2 become:

THEOREM 3. *When $f_y\equiv f_x+f_z$, the locus of $f(x, y, z)=0$ is degenerate.*

Note that in this case the definition of the degenerate locus still applies, with $U\equiv U_1\equiv 0$.

THEOREM 4. *If the locus of $f(x, y, z)=0$ is a curve, then $f_y\equiv f_x+f_z$.*

7. Orthogonal axes. When once we realize that the methods thus far developed allow axes to coincide without loss of their usefulness *as separate axes*, the way is opened for a drastic improvement in our system. On the n-axes plane we merely make the positive directions of separate axes 90° apart. Then, if more than four axes are involved, the fifth axis coincides with the first, the sixth with the second, *etc.* In this way the simple lattice points of plane analytic geometry are retained for any number of variables.

The equations of transformation from the two-axes to the n-axes systems [(7), p. 256 in I] now yield the following for the cases of three, four, and five

axes respectively (using x, y, z, u, and v for the axes in that order).

$$X = x - z; \qquad Y = y. \tag{16}$$

$$X = x - z; \qquad Y = y - u. \tag{17}$$

$$X = x - z + v; \qquad Y = y - u. \tag{18}$$

The generalization to the axes x_1, y_1, x_2, y_2, $\cdots$ is obviously

$$X = x_1 - x_2 + x_3 - \cdots; \qquad Y = y_1 - y_2 + y_3 - \cdots. \tag{19}$$

Evidently the G.C. of the point (X, Y) are, in cases (16), (17) and (18), $(x, Y, x-X)$, $(x, y, x-X, y-Y)$, and $(x, y, z, y-Y, X-x+z)$, where the values of the small letters are arbitrary. Thus there are $n-2$ parameters, or degrees of freedom, in the coordinates of the point (X, Y) on the n-axes plane. A function U of n variables becomes, over the point, a function of $n-2$ variables and the constants X and Y. If we equate to zero its partial derivative with respect to each of the variables, solve the $n-2$ equations simultaneously, substitute the results in the expression for U, and replace the latter by Z, we have the equation $Z=F(X, Y)$ of the bounding surface of the solid locus of U, if such exists.

8. Special theorems and functions. With the understanding that henceforth all axes are to be placed orthogonally, we may now combine and restate Theorems 1 and 2 for the case of four variables.

THEOREM 5. *When*

$$U_x + U_z \equiv U_y + U_u \equiv 0 \tag{20}$$

the locus of $U=f(x, y, z, u)$ *is degenerate; while if the locus is a surface, equations* (20) *hold.*

The method is to note that if $U=f(x, y, z, u)=F(X, Y)=F(x-z, y-u)$, then $U_x=U_X(1)$, $U_z=U_X(-1)$, *etc.* For three variables the test is simply: $U_x+U_z\equiv 0$; for five variables, $U_x+U_v-2U_z\equiv U_y+U_u\equiv 0$. The extension to n variables is simple.

To illustrate, the locus of

$$U = x^2 - y^2 + z^2 - u^2 - 2xz + 2yu \tag{21}$$

is degenerate, being the hyperbolic paraboloid $Z=X^2-Y^2$.

The locus of

$$U = \frac{x}{a} + \frac{y}{b} + \frac{z^2}{c^2} + \frac{u^2}{d^2} \tag{22}$$

is all space above and including the plane

$$Z = \frac{X}{a} + \frac{Y}{b} - \frac{c^2}{4a^2} - \frac{d^2}{4b^2}.$$

9. Generalizations of the paraboloid and ellipse. The following rather inclusive theorem will perhaps suggest why the writer believes, in his optimistic moments, that this particular extension of plane and solid analytic geometry is more in the spirit of those subjects, as well as more useful for some purposes, than the algebraic methods which stem from the abstract concept of "n-space," and which touch geometry mainly by definition and analogy.

THEOREM 6. *The locus of the terminating equation*

$$U + 1 = \frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} + \frac{x_2^2}{a_2^2} + \cdots \tag{23}$$

when the right side consists of three or more terms, is the filled elliptic paraboloid whose lower surface is

$$Z + 1 = \frac{X^2}{a_1^2 + a_2^2 + a_3^2 + \cdots} + \frac{Y^2}{b_1^2 + b_2^2 + b_3^2 + \cdots}. \tag{24}$$

When $U = Z = 0$, the plane locus of (23) *becomes the filled ellipse whose boundary is* (24).

The proof is sketched below. We note that, when n is even,

$$X = x_1 - x_2 + x_3 - \cdots + x_{n-1} - x_n. \tag{25}$$

When the value of x_n from (25) is substituted in (23) we find, upon differentiating, that

$$\frac{1}{2} U_{x_i} = \frac{x_i}{a_i^2} + (-1)^{i+1}(x_1 - x_2 + \cdots + x_{n-1} - X). \tag{26}$$

It may be verified that (26) is still true when n is odd. But each of the $n-1$ expressions for the right side of (26) becomes zero when

$$x_i = \frac{(-1)^{i+1}X}{A},$$

where $A = \sum_{i=1}^{n} a_i^2$. And when this value for x_i, together with a similar one for y_i, is substituted in (23) we get (24).

10. Further applications. In the case of two simultaneous equations in two unknowns the algebra of the solution usually takes the dominant role, and the intersections of the curves are merely incidental checks. But when more than two variables are involved, this situation is often reversed. For example, with the geometric background and aids the algebraic processes in the solution of four linear equations in four unknowns are reduced in the general case to the solution of four pairs of equations in two unknowns. Many standard form loci can be derived once for all, and when the loci do not touch there are no real solutions. But elaboration of these ideas cannot be included in this paper.

MATHEMATICAL NOTES

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Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON THE SIGN OF A FUNCTION IN A CLOSED INTERVAL

P. G. HODGE, JR., University of California, Los Angeles

A problem which arises frequently in applied mathematics is to show that a given function of one variable is always positive in a given interval. If the function is entirely known, this may be done, of course, by showing that it is positive at the endpoints and has no roots in the interval. However, this is not always feasible in the case where the function contains a parameter. In the present note a method is presented which may ease the computations considerably in certain cases. The related problems of showing that the function is always non-negative, always negative, or always non-positive, may also be solved by obvious adaptations of the method.

Consider a twice continuously differentiable function $F(x)$ defined in the interval $a \leqq x \leqq b$, where $F(a) > 0$, $F(b) > 0$. If $F(x)$ is non-positive at some interior point of $[a, b]$, it must have a relative minimum which is equal to or less than zero. Let $F(x_0)$ be such a minimum. Then the following relations are all valid at x_0:

$$F(x_0) \leqq 0, \qquad F'(x_0) = 0, \qquad F''(x_0) \geqq 0. \tag{1}$$

Now let $q(x)$ be non-negative in $[a, b]$, and let $p(x)$ be arbitrary. Define the *test function*:

$$T(x) = F(x) + p(x)F'(x) - q(x)F''(x). \tag{2}$$

The substitution of (1) into (2) shows that $T(x_0) \leqq 0$. Therefore, if it can be shown that the test function is positive for all $a < x < b$ it will follow that the hypothesis is false and therefore $F(x) > 0$ for all $a \leqq x \leqq b$. The test is a weak one, in that the converse is not necessarily true. If $T(x) \leqq 0$ for some x, no conclusion can be drawn as to the sign of $F(x)$

As a first example, consider the function

$$F(x) = (x - \alpha)e^x + e^{-\beta x} \tag{3}$$

in the interval $0 \leqq x \leqq 1$. Then

$$F(0) = 1 - \alpha, \qquad F(1) = (1 - \alpha)e + e^{-\beta},$$

which are both positive provided $\alpha < 1$. The test function for this case is

$$\begin{aligned} T(x) = {} & x\,(1 + p - q)e^x + [(1 - \alpha)p - (2 - \alpha)q - \alpha]e^x \\ & + (1 - \beta p - \beta^2 q)e^{-\beta x}. \end{aligned} \tag{4}$$

Constants p and q can be determined so that the coefficients of the last two terms

vanish:

(5) $p = (\alpha\beta^2 + 2 - \alpha)/\Delta, q = (1 - \alpha - \alpha\beta)/\Delta$, where $\Delta = (1 - \alpha)\beta^2 + (2 - \alpha)\beta$.

The corresponding test function is

$$T(x) = x(1 + \beta)^2 e^x/\Delta. \tag{6}$$

For the method to yield results, $T(x)$ must be positive and $q(x)$ non-negative. It follows from (5) and (6) that $\Delta > 0$ and $1 - \alpha - \alpha\beta \geqq 0$. Sufficient conditions for the function (3) to be always positive in $[0, 1]$ are thus seen to be

$$\alpha < 1, \qquad \beta > 0, \qquad \alpha(1 + \beta) \leqq 1. \tag{7}$$

The method is particularly valuable when $F(x)$ is a polynomial. In this case, take

$$p(x) = Ax + B, \qquad q(x) = (Cx + D)^2. \tag{8}$$

If $F(x)$ is a polynomial of degree n, the test function will also be a polynomial of degree n. In general, the four constants A, B, C, D may then be determined so that the coefficients of the four highest terms vanish, so that $T(x)$ becomes of degree $n-4$. In practice, this will be somewhat laborious, and it may be more economical to set some of the constants equal to zero and reduce the polynomial by a smaller degree.

As an example, consider the fourth degree polynomial

$$P_4(x) = x^4 - 3\lambda x^3 + 4\lambda x^2 - 2\lambda^2 x + 1 \tag{9}$$

in the interval $0 \leqq x \leqq \infty$, where $0 \leqq \lambda \leqq 1$. Take $q(x) = 0$, $p(x) = Ax + B$, so that

$$\begin{aligned} T(x) = {} & (1 + 4A)x^4 + (-3\lambda - 9\lambda A + 4B)x^3 + (4\lambda + 8\lambda A - 9\lambda B)x^2 \\ & + (-2\lambda^2 - 2\lambda^2 A + 8\lambda B)x + (1 - 2B\lambda^2). \end{aligned} \tag{10}$$

To reduce $T(x)$ to a quadratic, set $A = -1/4$, $B = 3\lambda/16$. Then

$$T(x) = (1/16)[(32\lambda - 27\lambda^2)x^2 + (16 - 6\lambda^3)]. \tag{11}$$

Since this is greater than zero for all $0 \leqq \lambda \leqq 1$ and for all x, the given polynomial is positive for all positive x.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

AN EXTENSION OF PLANE ANALYTIC GEOMETRY

J. A. WARD, University of Kentucky

An interesting sidelight may be shown to plane analytic geometry by letting one of the variables, say y, assume imaginary values. To this end replace the Y-axis by the complex UV-plane perpendicular to the X-axis to give three mutually perpendicular axes with the XU-plane identified as the conventional XY-plane. Then $x=a$, $y=b+ci$ may be written $x=a$, $u=b$, $v=c$ and is represented as the point (a, b, c) in space, and conversely. The "semi-complex" graph of a real function, $f(x, y)=0$, will consist of the locus of all points (a, b, c) such that $x=a$, $y=b+ci$ satisfy the equation.*

In $x^2-y^2=a^2$, $y=\pm\sqrt{x^2-a^2}$ is real for $x^2\geqq a^2$. Therefore $u=\sqrt{x^2-a^2}$, $v=0$ is the usual equilateral hyperbola in the XU-plane. This is the real portion of the curve. For $x^2<a^2$, $y=\pm i\sqrt{a^2-x^2}$ is pure imaginary and $v=\pm\sqrt{a^2-x^2}$, $u=0$, is a circle in the XV-plane, which is the imaginary portion. (See figure 1.) Thus the "semi-complex" graph of $x^2-y^2=a^2$ is a curve in two perpendicular planes. Notice that the circle in the XV-plane connects the (real) vertices of the hyperbola so that the curve is now all in one piece.

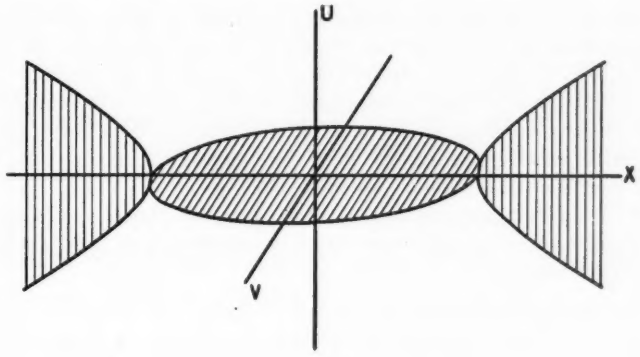

FIG. 1

Moreover if $y^2=f(x)$ is any real function of x, $u=\pm\sqrt{f(x)}$, $v=0$ will give the real portion and $v=\pm\sqrt{-f(x)}$, $u=0$ will give the imaginary portion of the "semi-complex" graph. The non-excluded portions of the real and imaginary parts of the planes complement each other to give a single connected whole, so that the real and imaginary portions of the graph intersect at right angles on the

* See also, A. J. Kempner, Bulletin A. M. S., vol. 41, 1935, pp. 809–843.

X-axis at the boundaries of the excluded portions of the real (or imaginary) planes.

Certain "mysteries" of other curves become clear when studied from this standpoint. The real curve of $y^2=(x-1)^2(x-2)$ has an isolated point at (1, 0). The reason for this is obvious in the "semi-complex" graph (figure 2), for we see the isolated real point is located where the loop in the imaginary part closes and is no longer isolated.

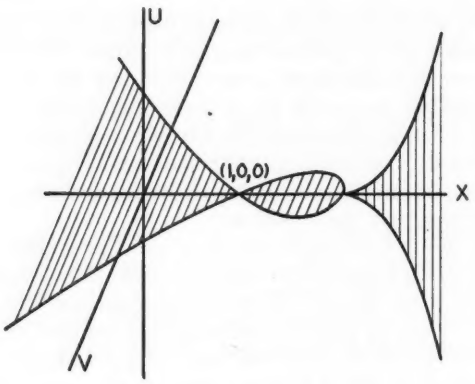

FIG. 2

"Semi-complex" graphs are not always as simple as those given above. If in the real equation $f(x, y)=0$ we let $y=u+iv$ (u and v real) we get $g(x, u, v)+iv\cdot h(x, u, v)=0$. Hence the "semi-complex" graph is the intersection of the real surfaces $g(x, u, v)=0$ and $v\cdot h(x, u, v)=0$. Since the latter function factors, the graph in space consists of two parts: $g(x, u, 0)=0$ (which is the usual graph $f(x, y)=0$ with y real) and intersection of the surfaces $g(x, u, v)=0$ and $h(x, u, v)=0$ determined by the imaginary values of y.

For $y^2+(ax+b)y+f(x)=0$, $h(x, u, v)=0$ becomes the plane $2u+ax+b=0$. Thus the semi-complex graph of any integral quadratic function of y lies in two perpendicular planes.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1026. *Proposed by Michael Golomb, Purdue University*

Orders have been given in a certain country that from now on all scientific writing shall be done in a secret transcription, consisting of a permutation of the alphabet of the country. A foreigner, unacquainted with the language and its alphabet is given the privilege of seeing the transcribed form of any five letters he may designate. He wishes to learn the common form of the word *atom*, which he is told has five letters. How can he manage this, if reference to the inverse transcription is ruled out?

E 1027. *Proposed by Ruth Clark and Robert Oeder, Los Alamos Scientific Laboratory*

Find the total area enclosed by the set of circles formed as follows: Construct the inscribed circle of a triangle ABC and draw the tangent to this circle which is parallel to a selected side of the triangle; construct the inscribed circle of the new triangle so cut off and draw the tangent to this circle which is parallel to the selected side of triangle ABC; repeat the process indefinitely. Also show that the set of maximum area is obtained if the lines are drawn parallel to the shortest side of triangle ABC.

E 1028. *Proposed by Vern Hoggatt, Oregon State College*

Do there exist Pythagorean triangles whose sides are Fibonacci numbers?

E 1029. *Proposed by A. E. Currier, United States Naval Academy*

Sum the series $1+1/3+2!/(3)(5)+3!/(3)(5)(7)+\cdots$.

E 1030. *Proposed by Charles Salkind, Polytechnic Institute of Brooklyn*

Consider the polygon formed by the internal trisectors of the angles of a given n-gon, intersecting in neighboring pairs.

(1) Prove that a necessary and sufficient condition that the trisector polygon be regular is that the parent n-gon be regular when $n \geqq 4$.

(2) Prove that the area ratio between the parent and trisector polygons is always irrational.

SOLUTIONS

An Inequality for the Triangle

E 996 [1952, 41]. *Proposed by Josef Andersson, Vaxholm, Sweden*

If a, b, c and r, r_a, r_b, r_c denote the measures of the sides and of the radii of the inscribed and escribed circles of a triangle, show that

$$abc/r \geq a^3/r_a + b^3/r_b + c^3/r_c.$$

Solution by C. V. Fronabarger, Southwest Missouri State College. If the desired relation is true, then, clearing of fractions and making use of the relations $rr_ar_br_c = r^2s^2$, $r_ar_b = s(s-c)$, $r_br_c = s(s-a)$, $r_cr_a = s(s-b)$, where s is the semi-perimeter, we get, after division by rs,

$$abcs \geq a^3(s-a) + b^3(s-b) + c^3(s-c),$$

which may be written as

$$\phi \equiv a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geq 0.$$

But, obviously, $\phi = 0$ when $a = b = c$ and $\phi > 0$ when a, b, c are not all equal. Hence, since the steps may be reversed, we may conclude that the desired relation is true.

Also solved by Ruth Clark, Tadashi Kishi and Robert Oeder (jointly), M. S. Klamkin, L. A. Ringenberg, Azriel Rosenfeld, Roscoe Woods, E. J. Zirkel, and the proposer.

Some Inequalities Related to Fermat's Last Theorem

E 997 [1952, 41]. *Proposed by David Ellis, University of Florida*

Let x, y, z, n be positive integers such that $n > 1$ and $x^n + y^n = z^n$. Show that

$$z^{n-1} < \min(x^n, y^n) < \max(x^n, y^n) < z^n, \tag{1}$$

$$1/z < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < z. \tag{2}$$

Solution by R. Kissling, University of California. Since $2^{1/n}$, $n > 1$, is irrational, we can suppose that $z > x > y \geq 1$. Then

$$\begin{aligned} y^n &= z^n - x^n \geq z^n - (z-1)^n \\ &= z^{n-1} + z^{n-2}(z-1) + \cdots + (z-1)^{n-1} > z^{n-1}. \end{aligned}$$

Thus $\min(x^n, y^n) > z^{n-1}$. That $\max(x^n, y^n) < z^n$ is obvious. This disposes of (1).

Again,

$$y^n = z^n - x^n \geq (x+1)^n - x^n > nx^{n-1},$$

whence $(y/x)^n > n/x$, and consequently $(x/y)^n < x/n$. Thus we have

$$n/x < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < x/n,$$

which implies (2) but is considerably sharper.

Also solved by Herbert Emich, Vern Hoggatt, M. S. Klamkin, Kovina Milosevich, Azriel Rosenfeld, W. L. Strother, and the proposer.

Maximum Area in a Corner

E 998 [1952, 41]. *Proposed by Leo Moser, University of Alberta*

Show that the largest area A that can be enclosed by two fixed lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by

$$A = (n/4) \cot (\theta/2n).$$

Solution by Vern Hoggatt, Oregon State College. We attain the desired result through the following sequence of theorems.

THEOREM 1. *Among all the curves of given length joining two given points A and B, the one which, together with the chord AB, bounds the maximum area is a circular arc.*

See, *e.g.*, Bolza, *Lectures in the Calculus of Variations*, ex. 13, p. 210.

THEOREM 2. *The triangle AOB having the maximum area subject to the conditions* $AB = m$ *and* $\sphericalangle AOB = \alpha$ *is the one in which* $OA = OB$.

For of all triangles having a common base and equal opposite angles, the isosceles triangle has the maximum altitude.

THEOREM 3. *Among all the curves of given length s whose endpoints A and B lie on two half-lines* OL_1 *and* OL_2, *forming an angle* $\theta \leqq \pi$, *the one which, together with the half-lines, bounds the maximum area* (assuming such a maximizing curve exists) *is a circular arc having its center at O.*

Suppose there exists a maximizing curve C of length s. Then C must be a circular arc, since otherwise the area between C and the chord AB could be increased by theorem 1, and the triangle AOB must be isosceles, since otherwise the area of triangle AOB could be increased by theorem 2. Thus C is a circular arc with its center on the bisector of angle AOB. We now determine the radius r so that the area will be a maximum. If ϕ is the central angle of the arc, the area of the figure is found to be

$$r^2(\phi - \sin \phi)/2 + (r \sin \phi/2)^2 \cot \theta/2.$$

Setting $r = s/\phi$, differentiating with respect to ϕ, setting the derivative equal to zero, and solving for $\cot \theta/2$, we find

$$\cot \theta/2 = \frac{\phi(1 + \cos \phi) - 2 \sin \phi}{\phi \sin \phi - 2(1 - \cos \phi)} = \frac{\sin \phi}{1 - \cos \phi} = \cot \phi/2.$$

Thus, for maximum area, $\phi = \theta$, or $r = OA$.

THEOREM 4. *The maximum area A that can be enclosed by two fixed half-lines, inclined at an angle* $\theta \leqq \pi$, *and n straight line segments of unit length is that given*

by the situation where the n unit line segments occur as bases of n equal isosceles triangles having a common vertex at the intersection of the two fixed half-lines.

Draw the circular arc C, having the intersection of the half-lines as center, and passing through the extremities of the n unit segments when arranged as described above. Then, if u denotes the area of a circular segment bounded by C and one of the unit segments, and if r denotes the radius of C, we have

$$A + nu = r^2\theta/2.$$

Suppose, now, that the area A' of some other polygon bounded by the two half-lines and n unit segments exceeds the area A. On the outward side of each of the n unit segments place one of the circular segments of area u considered above. We now have a figure bounded by the two half-lines and n circular arcs and having an area

$$A' + nu > r^2\theta/2.$$

Since we now have a contradiction of theorem 3, our present theorem is established.

THEOREM 5. *The maximum area A that can be enclosed by two fixed half-lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by*

$$A = (n/4) \cot (\theta/2n).$$

By theorem 4 the maximum area consists of n equal isosceles triangles with unit bases and with vertex angles equal to θ/n. The altitude of each triangle is $\frac{1}{2} \cot (\theta/2n)$, and the area of each triangle is $\frac{1}{4} \cot (\theta/2n)$, whence the total maximum area is $(n/4) \cot (\theta/2n)$.

Also solved by Julian Braun, E. I. Gale, R. D. Oeder and Tadashi Kishi (jointly), and C. S. Ogilvy. Most of these solutions assumed theorem 4 above.

As the proposer pointed out, if $\theta = \pi/m$, where m is a positive integer, the result follows easily from a well known extremal property of the regular polygon of $2mn$ sides. The particular case $\theta = \pi$, $n = 3$, is the familiar 'gutter' problem found in many calculus texts.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4498. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

Prove that every complex number is a zero of some entire function whose power series has rational coefficients.

4499. *Proposed by P. G. Federbush, Student, Rutgers University*

The coefficients of the congruence

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv 0 \pmod{p}$$

are subject to the restriction $0 \leqq a_j \leqq p-1$, $a_0 \neq 0$, but otherwise are taken at random. p is a prime. What is the probability that the congruence has no solution?

4500. *Proposed by J. R. Musselman, Western Reserve University*

Given three points A_i $(i=1, 2, 3)$ and a line L cut by the lines A_jA_k making angles α_i in the positive sense. Show that the lines drawn through A_i making angles $\pi-\alpha_i$, in the positive sense, with L are concurrent at a point P on the circumcircle of $A_1A_2A_3$. Further, if the altitudes from A_i make angles β_i with L, then the lines through A_i making angles $\pi-\beta_i$ with L are concurrent on the circumcircle of $A_1A_2A_3$ at a point Q diametrically opposite to P.

4501. *Proposed by C. D. Olds, San Jose State College, California*

Show that

$$\log (1.1010010001 \cdots) = \frac{1}{11} + \frac{1}{2}\,\frac{1}{101} + \frac{1}{3}\,\frac{1}{1001} + \cdots .$$

4502. *Proposed by Karl Goldberg, Brooklyn, New York*

Prove that two groups A and B are isomorphic if and only if there exist operations "$\cdot$" and "$*$" defined over the elements of the groups as follows:

$$a \cdot b \in A \qquad a \cdot (b^* a') = (a \cdot b)a'$$

$$b^* a \in B \qquad b^* (a \cdot b') = (b^* a)b'$$

where $a, a' \in A$; $b, b' \in B$.

SOLUTIONS

4421 [1951, 42]. *Correction*

In the May issue, p. 338, in the first equation in solution I, replace the summation sign by product sign.

An Application of Wolstenholme's Theorem

4428 [1951, 194]. *Proposed by C. D. Olds, San Jose State College, California*

If p is any prime number >3, prove that

$$_{kp-1}C_{p-1} \equiv {}_{jp-1}C_{p-1} \pmod{p^3},$$

where j, k are positive integers and ${}_nC_r$ is the number of combinations of n different things taken r together.

Solution by Ernest Trost, Technikum Winterthur, Zürich, Switzerland. From Fermat's theorem follows

$$\begin{aligned} x^{p-1} - 1 &\equiv (x-1)(x-2)\cdots(x-p+1) \\ &= x^{p-1} - s_1x^{p-2} + \cdots - s_{p-2}x + s_{p-1} \pmod{p}. \end{aligned}$$

Hence we have $s_i \equiv 0 \pmod{p}$, $(i=1, 2, \cdots, p-2)$, $s_{p-1}=(p-1)!$ and for $x=p>3$

$$(p-1)! \equiv s_{p-2}p^2 - s_{p-2}p + s_{p-1} \pmod{p^3},$$

whence

$$s_{p-2} \equiv ps_{p-3} \equiv 0 \pmod{p^2}.$$

Now, if $p>3$, we obtain, upon putting $x=kp$,

$$\prod_{i=1}^{p-1}(kp-i) \equiv s_{p-3}k^2p^2 - s_{p-2}kp + s_{p-1} \equiv (p-1)! \pmod{p^3}.$$

Therefore, upon dividing by $(p-1)!$, we have

$$_{kp-1}C_{p-1} \equiv 1 \pmod{p^3},\ k = 1, 2, \cdots,$$

which implies the desired result.

Also solved by A. N. Aheart, D. H. Browne, Leonard Carlitz, Margaret W. Maxfield, Leo Moser, J. H. Wahab, and R. E. Wild.

Editorial Note. Several solvers refer to Wolstenholme's and other relevant theorems. See Hardy and Wright, *An Introduction to the Theory of Numbers*, pp. 85–88. Carlitz finds a proof by J. W. L. Glaisher in the *Quarterly Journal of Mathematics*, vol. 31 (1900), pp. 20–21, 325, where also additional results are obtained, including the following: If $p \geqq 5$, and if B_{p-3} denotes a Bernoulli number in the even suffix notation, then

$$_{kp-1}C_{p-1} \equiv 1 - \tfrac{1}{3}(k^2-k)p^3B_{p-3} \pmod{p^4}.$$

A Summation Problem

4431 [1951, 195]. *Proposed by M. S. Klamkin, Brooklyn Polytechnic Institute, New York*

If $S_n = \sum_1^n 1/r$, prove:

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}.$$

Solution by Robert Steinberg, University of California, Los Angeles. It suffices to prove the first and third expressions equal since the obvious identity

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}$$

implies the second. Now we have

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{r=1}^{n} \frac{1}{r} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{n+r} \right)$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \left(\frac{1}{r(n+r)^2} + \frac{1}{n(n+r)^2} \right) = 2\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{r(n+r)^2}$$

by symmetry. Upon setting $r = R$ and $r+n = N+1$, with appropriate changes in the limits of summation, this last form becomes $2\sum_1^{\infty} S_N/(N+1)^2$ and the proof is complete.

An interesting corollary may be obtained as follows. We have

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{1}{n^2 r}.$$

If the indicated summation in the (r, n)-plane is carried out along rational rays through the origin, the first integral point on such a ray is a point (R, N) such that R and N are relatively prime and all other points on this ray are the integral multiples of (R, N). Thus the contribution of this ray to the sum is

$$\frac{1}{N^2 R} \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

The contribution of all rational rays with $0 < R \leqq N$ is thus

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sum{}' \frac{1}{N^2 R}, \tag{1}$$

where the inner summation is to be made over all pairs of integers R and N such that $0 < R \leqq N$ and R and N are relatively prime. Since the expression (1) is equal to $2\sum_1^{\infty} 1/n^3$, we get the desired result:

$$\sum{}' \frac{1}{N^2R} = 2.$$

Also solved by T. M. Apostol, D. H. Browne, O. E. Stanaitis, J. Vales and A. Petracca, and the Proposer.

Editorial Note. Vales and Petracca obtain the following generalizations. Put

$$S(r, n) = \sum_{i=1}^{n} \frac{1}{i^r}, \qquad K(r, n) = \frac{1}{n+r-1} \sum_{i=1}^{n} K(r-1, i),$$

with $K(1, n) = 1/n$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^{h+2}} = \sum_{n=1}^{\infty} \sum_{i=1}^{h} \frac{S(i, n)}{(n+1)^{h+2-i}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{h+1}} = \sum_{n=1}^{\infty} \frac{K(h, n)}{h+n-1}.$$

Browne finds

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{S_n}{(n+1)^3} = 5 \sum_{n=1}^{\infty} \frac{S_n}{(n+1)^3},$$

but further extensions do not assume so neat a form.

A Circle Tangent to the Nine-Point Circle

4432 [1951, 195]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the midpoints of the sides of a triangle as centers, circles are described passing through the feet of the corresponding altitudes. Show that the circle orthogonal to these three circles is tangent to the nine-point circle of the given triangle at a point such that its distance from the foot of one altitude is equal to the sum of the distances from the feet of the other two altitudes.

Solution by O. J. Ramler, Catholic University, Washington, D. C. Let A, B, C be the midpoints of the sides of the triangle $A'B'C'$, and let the feet of the altitudes of $A'B'C'$ be D, E, F. Draw the circles (A), (B), (C) with centers A, B, C and radii AD, BE, CF respectively. Draw the common chords of the circumcircle (O) of ABC and each of the circles (A), (B), (C). Let X, Y, Z be the vertices of the triangle formed by these common chords. Then the incircle of XYZ is the circle mutually orthogonal to circles (A), (B), (C).

Let the common chord of circles (A) and (O) meet (O) again in D'. Then angle $AD'D$ = angle $ADD' = B - C$, since AD is parallel to BC. Hence YZ is antiparallel to BC with respect to AB and AC. Similarly XZ, the common chord of (B) and (O), is antiparallel to AC with respect to AB and CB. Hence ZX and ZY form an isosceles triangle with AB. The radical axis of circles (A) and (B) therefore bisects angle Z of triangle XYZ. From this it follows readily that the radical center of circles (A), (B), (C) is the incenter of triangle XYZ.

Let the incircle (I) of triangle XYZ touch the sides YZ, ZX, XY at L, M, N respectively. With respect to circle (A) the side YZ inverts into circle (O). Since

the incircle (I) is tangent to YZ at L, (I) must touch (O) where AL meets (O); call this point T. Point T is unique, that is, T is the point where M inverts with respect to circle (B), and N inverts with respect to (C). Moreover, since L and T are inverse with respect to (A), (I) and (A) are orthogonal. Similarly (I) is orthogonal to (B) and (C). Circles (I) and (O) are therefore tangent at T, and T is the center of similitude of (I) and (O). Hence triangles ABC and LMN are homothetic with respect to T so that

$$TL/TA = TM/TB = TN/TC = r/R$$

where r, R are the radii of (I) and (O). From $TL/TA = r/R$ and $AL \cdot AT = AD^2$ we obtain $TL \cdot AL = (r/R)AD^2$. Similarly we have $TN \cdot CN = (r/R)CF^2$ and thence $(TL \cdot AL)/(TN \cdot CN) = AD^2/CF^2$. From $TL/TN = AL/CN$ and $TL/TA = TN/TC$ it follows that

$$AT \cdot CF = TC \cdot AD. \tag{1}$$

Now $ACET$, $ACFT$ and $ACDT$ are cyclic quadrangles. Ptolemy's theorem provides the equations

$$ET \cdot AC = AT \cdot CE + AE \cdot TC, \tag{2}$$

$$FT \cdot AC = AT \cdot CF + AF \cdot TC, \tag{3}$$

$$DT \cdot AC = AT \cdot CD - AD \cdot TC. \tag{4}$$

Adding (3) and (4) and subtracting (2) we have

$$AC(DT + FT - ET) = AT(CD + CF - CE) + TC(AF - AD - AE).$$

Since $AD \| BC$ and $BE \| AC$, it follows that $CD = CE = AB$, and $AF = AE = BC$. Hence

$$AC(DT + FT - ET) = AT \cdot CF - TC \cdot AD = 0$$

from (1). Since $AC \neq 0$, it follows that $TD + TF = TE$.

Also solved by W. E. Byrne and Joseph Langr.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Meaningful Mathematics, A survey course for college students. By H. S. Kaltenborn. New York, Prentice-Hall, Inc., 1951. 15+397 pages. $4.75.

This is a text for a one year "cultural" type of survey course for students whose mathematical background may be limited. It aims to help the student gain insight into underlying ideas and processes as well as appreciation of the

role of mathematics in civilization.

A wide range of topics is treated: the nature and value of mathematics; number and computation; function and graph; algebra; numerical and analytic trigonometry; analytic geometry; differential and integral calculus. A chapter on geometry includes discussions of the nature of the postulational approach and of the relation of Euclidean to non-Euclidean geometries. A chapter on consumer mathematics features a more realistic discussion of installment buying than is usually found.

The chapters on calculus neglect almost all derivations, the connection of the indefinite with the definite integral being asserted without even being made plausible, yet these chapters treat a wide variety of applications: extrema; related rates; differentials; area; volume; work; motion; exponential growth; integration of series; *etc.* Throughout the book, the student is frequently offered a taste of a new mathematical morsel, which is snatched away before a good bite can be had—perhaps a necessary caution in that it might prove indigestible to a stomach unused to such fare.

Statements or discussions which may mislead the reader occur on pages 160, 169, 175, 305–306, 351. The explanation of a variation of "Nim" on pages 22–23 is faulty. The proof on page 298 can be simplified. The description of typical trigonometric problems on pages 195–197 might well be supplemented by scale construction suggestions.

Instructors who have tired of survey texts which conceal standard topics under fancy names and which strive to capture student interest by sparkling diction and bright jests, may welcome this book. It retains customary terminologies, and uses familiar types of examples and illustrations. On the whole, the presentation is competent, the writing simple and lucid, the format attractive. There is an answer section, and brief log-trig tables. Many of the exercises request brief written or oral reports on special mathematical topics, case histories, recreations, *etc.* Such sidelights are also stressed in the text proper.

It is difficult to evaluate the gain to the student from his skimming the surface of so much mathematical material. Perhaps he may occasionally delve into the richer strata with the help of the oral reports, which Professor Kaltenborn feels form a vital part of the course.

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NEW BOOKS RECEIVED

Industrial Process Control by Statistical Methods. By J. D. Heide. New York, McGraw-Hill Book Company, 1952. ix+297 pages. $6.00.

College Algebra. First Edition. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1952. xix+344 pages. $3.50.

The Elements of Mathematical Analysis. Second Edition. Volumes I and II. By J. H. Michell and M. H. Belz. New York, The Macmillan Company, 1952.

24+516 pages and 12+569 pages. $13.60 the set.

Brief Trigonometry. Revised Edition. By E. A. Cameron. New York, Henry Holt and Company, 1952. 6+151 pages. $2.10.

Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen. By Dr. Guido Hoheisel. Berlin, Germany, 1952, Walter de Gruyter and Company. 136 pages. DM 2,40.

Dictionary of Mind, Matter and Morals. By Bertrand Russell. New York, Philosophical Library, 1952. xiv+290 pages. $5.00.

Einführung in die freie Geometrie ebener Kurven. By Dr. Louis Locher-Ernst. Basel, Switzerland, Verlag Birkhäuser, 1952. 87 pages. Schw. Fr. 12.50.

Smithsonian Logarithmic Tables. By G. W. Spenceley, R. M. Spenceley, and E. R. Epperson. Washington, D. C., Smithsonian Institution. xii+402 pages, 1952. $4.50.

Econometrics. By Gerhard Tintner. New York, John Wiley and Sons, 1952. xiii+370 pages. $5.75.

Intermediate College Mechanics, A Vectorial Treatment. By D. E. Christie. New York. McGraw-Hill Book Company, 1952. xvi+454 pages. $7.00.

Elementary Differential Equations. By E. D. Rainville. New York, The Macmillan Company. xii+392 pages. 1952. $5.00.

Calculus. By A. H. Sprague. New York, The Ronald Press Company, 1952. xi+576 pages. $6.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB TOPICS

Bibliographies have been published for a considerable number of topics frequently used for club programs. A summary list may be of value to program directors and others.

The bibliographies which follow were published in this MONTHLY, the numbers referring to volume and page.

American mathematicians. *47*, 107.
Apportionment of representatives. *47*, 484; *49*, 115.
Arithmetical prodigies. *25*, 91.
The binary scale of notation, etc. *25*, 139.
Calculating machines. *43*, 99; *46*, 233; *47*, 106.
The cattle problem of Archimedes. *25*, 411.
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Codes and ciphers. *26*, 409.

Constructions with a double-edged ruler. **25**, 358.
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Fiedler's cyclography. **32**, 517.
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The following bibliographies appeared in *The Pentagon*.

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Solutions of the quadratic equation. 6, 66.
Squaring the circle. 9, 105.
The story of pi. 11, 34.
Trisection of an angle. 8, 21.
Women as mathematicians. 5, 67.

Several comprehensive bibliographies have been prepared by William L. Schaaf and published in *The Mathematics Teacher*. The following selections contain material for clubs.

The correlation of mathematics and science. 44, 340. (1. Curriculum integration in mathematics and science. 2. Biology and mathematics. 3. Chemistry and mathematics. 4. Physics and mathematics. 5. Mathematics and aviation. 6. Mathematics and navigation. 7. Miscellaneous applications of mathematics to science.)

Guidance: the case for mathematics. 44, 130. (1. Why learn mathematics? 2. Guidance in mathematics. 3. Functional competence in mathematics. 4. Mathematics for industry, trade, and business. 5. Mathematics for professional careers. 6. Cultural values of mathematics.)

Laboratory mathematics. 44, 422. (1. Laboratory methods and equipment. 2. Laboratory devices for plane geometry. 3. Laboratory devices for solid geometry. 4. Models for polyhedrons. 5. Laboratory devices for algebra. 6. Laboratory devices for trigonometry. 7. Models for conic sections. 8. Paper folding: dissection of figures. 9. Linkages. 10. Mechanical construction of curves.)

Logarithms and exponentials. 45, 361. (1. History of logarithms. 2. Theory of logarithms. 3. Logarithms and exponential functions. 4. Teaching logarithms.)

Mathematics and art. 43, 423. (1. Art. aesthetics, architecture. 2. Ornament and design; pattern; geometric drawing. 3. Dynamic symmetry. 4. The golden section.)

Mathematics and engineering. 44, 54. (1. Mathematics and engineering education. 2. Engineering and mathematics. 3. Texts and reference works on mathematics for engineers.)

Mathematics and modern science. **43**, 294.

Mathematical plays and programs. **44**, 526. (1. Plays, skits, and pageants. 2. Programs for assemblies and mathematics clubs.)

Modern calculating machines. **45**, 110. (1. General articles. 2. Technical books. 3. Digital machines; electronic calculators. 4. Differential analyzers. 5. Harmonic analyzers and synthesizers. 6. Network analyzers. 7. Algebraic equation solvers.)

The numerical solution of equations. **44**, 204. (1. General approximation methods. 2. On the location of roots. 3. Miscellaneous methods of solving cubic and quartic equations.)

The theorem of Pythagoras. **44**, 585. (1. Proofs and discussions. 2. Pythagorean numbers: integral and rational right triangles. 3. Special cases of rational right triangles; Heronian triangles. 4. Pythagorean theorem in design. 5. Books and monographs.)

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

A REPORT OF THE GENERAL ASSEMBLY OF THE INTERNATIONAL MATHEMATICAL UNION

The first General Assembly of the new International Mathematical Union met in Rome, Italy, on March 6–8, 1952. Eighteen member countries were represented by delegates and, in addition, there were observers present from Spain and Poland. The members of the United States Delegation were as follows: Professor M. H. Stone, University of Chicago, Chairman of the Delegation; Professor Einar Hille, Yale University; Professor J. R. Kline, University of Pennsylvania; Professor Saunders MacLane, University of Chicago; and Professor G. T. Whyburn, University of Virginia.

There was a lively and profitable discussion of the future activities of the Union; for example, a committee was set up to study the possibility of publishing a world directory of mathematicians. Other committees are to deal with methods of decreasing the cost of mathematical publication and with methods of facilitating the international exchange of persons. Of greatest interest to the Association is the reestablishment of the International Mathematical Instruction Committee (IMUK). This committee, charged with the study on an international basis of the important and difficult problems of improving mathematical instruction at all levels, is to consist of Professors Behnke (Germany), Châtelet (France), Fehr (Switzerland), Kurepa (Yugoslavia) and Jeffery (Canada). It is hoped that auxiliary national committees will be set up to assist in this work.

The business of the General Assembly proceeded in the most friendly fashion, though it was delayed somewhat by the necessity of saying everything both in French and in English. The chairmanship of the meeting was rotated from

country to country. Three new countries—Pakistan, Spain and Yugoslavia—were admitted to membership, and it was the hope of all that still other countries will apply soon for membership. It was decided that both English and French shall be equally official languages for the work of the Union.

Officers of the Union were elected as follows: President, M. H. Stone (U.S.A.); Vice-presidents, E. Borel (France) and E. Kamke (Germany); Secretary, E. Bompiani (Italy); Members of the Executive Committee: W. V. D. Hodge (England), S. Iyanaga (Japan) and B. Jessen (Denmark).

As is often the case with such meetings, one of the most important features was the chance of talking personally with colleagues from other countries. These opportunities were extended by an excursion to Tivoli on March 9th and a two day session for mathematical papers on March 10–11. There were ten papers, from as many countries, full of interest. For my own part, the high point was the penetrating new results of H. Cartan on Hilbert's theorem on chains of syzygies.

SAUNDERS MACLANE

PERSONAL ITEMS

Professor J. Hobart Bushey of Hunter College was the delegate of the Association at the Convocation which was held at the United States Military Academy on May 20, 1952 in celebration of the sesquicentennial year of the Academy.

Professor Wayne Dancer of the University of Toledo represented the Association at the inauguration of President Ralph MacDonald of Bowling Green State University on May 10, 1952.

Professor L. M. Graves of the University of Chicago was the representative of the Association at the annual meeting of the American Council on Education in Chicago on May 2–3, 1952.

Professor T. L. Reynolds, Millsaps College, was the delegate of the Association at the program commemorating the seventy-fifth anniversary of Jackson College on May 1–2, 1952.

Professor V. F. Cowling of the University of Kentucky has been awarded a Ford Foundation Faculty Fellowship for the year 1952–1953; he is on leave of absence from the University of Kentucky and is located at Yale University.

Professor F. L. Griffin has been honored by the establishment of the F. L. Griffin Fund for the Improvement of Teaching by the alumni of Reed College; Professor Griffin has retired from his position as Professor of Mathematics at Reed College after forty-one years of service.

Professor J. S. Miller of Dillard University has been awarded a Ford Foundation Fellowship for 1952–1953.

Professor George Polya of Stanford University has been elected an honorary member of the Board of the Mathematical Society of France.

Professor Herbert Robbins, University of North Carolina, has been awarded a Guggenheim Fellowship and will spend the year 1952–1953 as a member of the

Institute for Advanced Study.

Kent State University announces the following promotions: Associate Professor Emerson Jenkins to a professorship; Assistant Professor Marvin Johnson to an associate professorship.

Louisburg College announces the following: Professor T. C. Amick, head of the Department of Mathematics, will retire at the end of the fall semester of 1952; Associate Professor Elizabeth Johnson will succeed Professor Amick as Head of the Department.

New Mexico College of Agriculture and Mechanic Arts makes the following announcements: Assistant Professor Max Kramer has been promoted to an associate professorship; Dr. O. B. Ader, formerly of Mississippi Southern College, has been appointed to an instructorship.

Skidmore College announces: Associate Professor Mary E. Williams is on leave of absence during the first semester of 1952–1953 and is studying at Cambridge University; Professor Emeritus C. H. Forsyth of Dartmouth College is teaching at the college during her absence.

Swarthmore College announces the retirement of Professor Arnold Dresden and Professor R. W. Marriott; Professor Dresden has been a member of the faculty for twenty-five years, Professor Marriott has taught at the college for forty-six years.

Western Michigan College reports the promotion of Assistant Professors H. H. Hannon, J. K. Peterson, and C. S. Sams to associate professorships.

Dr. Jerome Blackman has been appointed to an instructorship at Syracuse University.

Dr. Archie Blake of the Mechanical Research Corporation, Chicago, Illinois, has accepted a position as Head of the International Business Machines Computing Section of the Cornell Aeronautical Laboratory, Buffalo, New York.

Instructor R. G. Bradshaw of Clarkson College has been promoted to an assistant professorship.

Dr. H. F. Bright, formerly of the University of Texas, has accepted a position as Chief, Technical Services, Human Resources Research Office, George Washington University.

Mr. H. H. Brown, previously of the Armour Research Foundation, Illinois Institute of Technology, has accepted a position as Senior Research Engineer with the Franklin Institute Laboratories for Research and Development, Philadelphia, Pennsylvania.

Associate Professor Kenneth E. Brown of the University of Tennessee has been appointed Specialist for Mathematics in the United States Office of Education.

Professor Emeritus W. B. Carver of Cornell University was Visiting Professor at Southern Illinois University during the 1952 Summer Session.

Mr. S. H. Chasen who has been a graduate student at Emory University has a position as Mathematician at the United States Naval Air Testing Station, Patuxent River, Maryland.

Mr. M. L. Coffman, East Texas State Teachers College, has been appointed to an instructorship in the Department of Physics of Texas Agricultural and Mechanical College.

Dr. G. F. Cramer of the Defense Department has accepted a position as Engineer with R. C. A. Victor, Camden, New Jersey.

Assistant Professor Albert Edrei of the University of Colorado has been appointed to an associate professorship at Syracuse University.

Mr. Rudolf Feige, formerly of the University of Cincinnati, has a position as Electrical Engineer with the American Can Company, Cincinnati, Ohio.

Mr. A. R. Friedenheit, previously a student at Washington and Jefferson College, is now a stress analyst with North American Aviation, Columbus, Ohio.

Associate Professor Fritz Herzog of Michigan State College has been promoted to a professorship.

Dr. T. J. Jaramillo who has been Chairman of the Department of Mathematics and Professor of Mathematics at Far Eastern University, Manila, Philippine Islands, for the last four years has returned to the Engineering Mechanics Division of Armour Research Foundation, Illinois Institute of Technology, as Senior Scientist.

Professor E. P. Lane of the University of Chicago has retired.

Dr. R. A. Leibler of the Navy Department has accepted a position as Mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. H. G. Mazurkiewicz, previously of the Willow Run Research Center, University of Michigan, is now a group engineer, electronics, with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. Jack Minker who has been a dynamics engineer at Bell Aircraft Corporation, Niagara Falls, New York, is now an engineer in the Advanced Development Section, Radio Corporation of America, Camden, New Jersey.

Miss Florence Moline of Union College, Lincoln, Nebraska, has been promoted to an assistant professorship.

Mr. Robert E. Montgomery is a mathematical research analyst with the Sandia Corporation, Albuquerque, New Mexico; he is also a part-time faculty member of the Department of Mathematics and Astronomy of the University of New Mexico.

Dr. W. K. Moore of the University of Kansas has been appointed to an assistant professorship at Albion College.

Mr. C. R. Morris, formerly of the United States Navy Hydrographic Office, Suitland, Maryland, has accepted a position with the Geotechnical Corporation, Dallas, Texas.

Dr. Paul Nesbeda of Catholic University of America has a position as Mathematician with the Radio Corporation of America, Camden, New Jersey.

Dr. John Reckzeh has been appointed to an associate professorship at State Teachers College, Jersey City, New Jersey.

Dr. F. Virginia Rohde of the University of Florida has been promoted to an assistant professorship.

Associate Professor M. F. Rosskopf of Syracuse University has been appointed to an associate professorship at Teachers College, Columbia University.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo has been promoted to an associate professorship.

Mr. Cecil Schwartz, formerly an engineer with the Armour Research Foundation, Illinois Institute of Technology, is now an engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. W. L. Shepherd of the University of Oregon has been appointed to an instructorship at Texas Western College of the University of Texas.

Assistant Professor J. C. Smith of Lafayette College has been promoted to an associate professorship.

Dr. Roscoe Stinetorf has been appointed Chairman of the Department of Mathematics of Wagner College.

Associate Professor R. R. Stoll of Lehigh University has been appointed to a professorship at Oberlin College; during the past summer Professor Stoll served as a consultant to the National Science Foundation.

Mr. N. D. Vlachos, formerly of the Kellett Aircraft Corporation, Camden, New Jersey, has accepted a position as Research Associate with the Hercules Powder Company, Allegany Ballistics Laboratory, Cumberland, Maryland.

Mr. R. F. Wheeling of Brown University is now a technologist with the Socony-Vacuum Oil Company, Paulsboro, New Jersey.

Professor A. E. Whitford of Alfred University has retired with the title of Professor Emeritus; Professor Whitford has been engaged in college teaching for fifty-one years.

Mr. W. F. Winn of the Armed Services Electro Standards Agency, Fort Monmouth, New Jersey, has been promoted to the position of Electronic Engineer.

Associate Professor Emeritus Joseph Barnett, Oklahoma Agricultural and Mechanical College, died on April 12, 1952. He had been a member of the Association for twenty-three years.

Mr. R. D. Bartz, an undergraduate student at the University of Wisconsin, died on February 14, 1951.

Professor M. C. Foster of Wesleyan University, Middletown, Connecticut, died on April 9, 1952.

Professor Marie Litzinger, chairman of the Department of Mathematics of Mount Holyoke College, died on April 7, 1952.

Mr. J. J. Newman, a student at Columbia University, died on September 10, 1951.

Professor Emeritus I. C. Nichols of Louisiana State University died on March 13, 1952. He was a charter member of the Association.

Mr. R. R. Shumway, a retired professor of the University of Minnesota, died on April 19, 1952. He was a charter member of the Association.

Professor Emeritus E. W. D. Taylor of Pomona College died on April 7, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1952 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	G. B. Price, University of Kansas
Missouri	R. J. Michel, Southeast Missouri State Teachers College
Ohio	I. A. Barnett, University of Cincinnati
Pacific Northwest	R. D. James, University of British Columbia
Southeastern	F. A. Lewis, University of Alabama
Southwestern	R. F. Graesser, University of Arizona
Upper New York State	C. W. Munshower, Colgate University
New England Region	B. H. Brown, Dartmouth College

As usual, great interest has been shown in each Section in the selection of a Governor to represent the members of that Section. The close contests in many Sections seemed to indicate that the members felt that the losing candidates as well as the successful ones were worthy of membership on the Board of Governors. Of the Sections voting this year the highest percentage of votes cast was 53% in the Kansas Section.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 130 persons have been elected to membership by the Board of Governors on applications duly certified.

M. J. AEGERTER, M.S. (Wisconsin) Grad. Student, University of Wisconsin, Madison, Wis.

M. A. A. AL-BASSAM, Ph.D. (Texas) Asst. Professor, Huston-Tillotson College, Austin, Tex.

A. G. ANDERSSON, Student, University of Alabama, University, Ala.

JOSEPH ANDRUSHKIW, Ph.D. (Ukrainian Free U.) Asso. Professor, Seton Hall University, South Orange, N. J.

P. L. ARMSTRONG, A.M. (Southwestern Presby. U.) Asst. Professor, Clemson College, S. C.

M. J. ARTHURS, Student, Harvard University, Cambridge, Mass.

G. B. AX, M.A. (Columbia) Asst. Professor, Virginia Military Institute, Lexington, Va.

J. L. BARNES, Ph.D. (Princeton) Professor of Engineering, University of California at Los Angeles, Calif.

MRS. MABEL S. BARNES, Ph.D. (Ohio State) Lecturer, Occidental College, Los Angeles, Calif.

PETER BEDROSIAN, Student, McMaster University, Hamilton, Ontario, Canada

R. J. BEEBER, M.S. (Catholic U.) Grad. Stu-

dent, Columbia University, New York, N. Y.

M. R. BOTTACCINI, B.A. (Southern California) Head, Torpedo Dynamics Section, Naval Ordnance Test Station, Pasadena, Calif.

EVELYN BOYD, Ph.D. (Yale) Asso. Professor, Fisk University, Nashville, Tenn.

B. J. BOYER, Student, Purdue University, Lafayette, Ind.

R. B. BREDEMEIER, B.A. (Walla Walla) Grad. Assistant, Oregon State College, Corvallis, Ore.

G. E. BREDON, Student, Stanford University, Calif.

W. R. BRITTENHAM, M.S. (Wisconsin) Grad. Student, University of Wisconsin, Madison, Wis.

J. C. CALDWELL, Ed.M. Instr., Edward Waters College, Jacksonville, Fla.

H. E. CAMPBELL, Ph.D. (Wisconsin) Asst. Professor, Emory University, Ga.

RUTH D. CLARK, M.A. (Wisconsin) Staff Member, Los Alamos Scientific Laboratory, N. M.

J. T. CLAUSEN, JR., B.S. (Beloit) Chief, Methods and Data Analysis, White Sands Proving Ground, Las Cruces, N. M.

C. L. COPE, M.A. (Georgia) Asst. Professor, University of Georgia, Atlanta, Ga.

WALTER CORNETZ, B.A. (N.Y.U.) Communications Engineer, Grumman Aircraft Engineering Corporation, Long Island, N. Y.

G. E. DECKER, M.A. (N.Y.S.C. for Teachers, Albany) Teacher, Lewiston High School, N. Y.

AUGUST DECKERT, B.A. (Washington S.C.) Mathematician, White Sands Proving Grounds, Las Cruces, N. M.

J. J. DENNIS, Ph.D. (Northwestern) Chairman, Department of Mathematics, Clark College, Atlanta, Ga.

A. W. DICKINSON, Student, University of Massachusetts, Amherst, Mass.

RAYMOND DOBY, M.S. (N.Y.U.) Development Engineer, Sylvania Electric Products, Kew Gardens, N. Y.

ALBERT EDREI, Dr.Sci.Math. (Zürich) Asst. Professor, University of Colorado, Boulder, Colo.

HARVEY EISENBERG, B.A. (Brooklyn) Mathematician, Ft. Monmouth, N. J.

J. M. ELKIN, M.A. (Columbia) Chief Statistician, Railroad Retirement Board, Chicago, Ill.

H. MARGARET ELLIOTT, Ph.D. (Radcliffe) Asst. Professor, Washington University, St. Louis, Mo.

G. V. EMERSON, Student, Hobart College, Geneva, N. Y.

A. M. FLEISHMAN, B.A. (Bridgewater) Mathematician, Naval Proving Ground, Dahlgren, Va.

S. A. FOOTE, Student, University of Alabama, University, Ala.

J. W. FRANKOWSKY, M.S. (N.Y.U.) Acting Chairman, Department of Mathematics, Lincoln University, Pa.

G. V. FRANKS, Student, Holy Cross College, Worcester, Mass.

A. R. FRIEDENHEIT, Student, Washington and Jefferson College, Washington, Pa.

P. W. GILBERT, Ph.D. (Duke) Asst. Professor, Syracuse University, N. Y.

T. L. GLAHN, B.S. (Oregon S.C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

REV. H. H. GOTTBRATH, Sc.M. (DePaul) Instr., Bellarmine College, Louisville, Ky.

C. C. GRANDY, Student, Colorado Agricultural and Mechanical College, Ft. Collins, Colo.

BERNARD GREENSPAN, M.A. (Brooklyn) Asst. Professor, Drew University, Madison, N. J.

J. D. HAGGARD, Ed.D. (Missouri) Asst. Professor, Kansas State Teachers College, Pittsburg, Kans.

LOIS S. HALE, Student, University of Georgia, Athens, Ga.

J. L. HAMMERSMITH, B.S. (Michigan) Mathematician, Naval Research Laboratory, Washington, D. C.

L. F. HAYWARD, Student, Southwestern College, Winfield, Kan.

H. S. HEAPS, M.A. (Toronto) Asst. Professor of Civil Engineering, Nova Scotia Technical College, Halifax, Nova Scotia, Canada

P. K. HENNEY, Student, Ball State Teachers College, Muncie, Ind.

R. E. HILL, Student, University of California, Berkeley, Calif.

M. H. HOEHN, M.A. (California) Instr., University of Idaho, Moscow, Idaho

J. H. HORNBACK, Ph.D. (Illinois) Asst. Professor, University of Alabama, University, Ala.

H. F. HUNTER, B.S. (Northwestern) IBM CPC Operator, Northrop Aircraft Company, Hawthorne, Calif.

R. T. IVES, Student, Haverford College, Pa.

R. E. JACKSON, Student, University of California at Los Angeles, Calif.

L. W. JACOBS, III, Student, Central College, Fayette, Mo.

E. H. JAKUBOWSKI, B.A. (American International) Physicist, U. S. N. Mine Countermeasures Station, Panama City, Fla.

H. H. JOHNSON, B.A. (San Jose S.C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

W. W. JONES, M.A. (Cornell) Asso. Professor, Kentucky State College, Frankfort, Ky.

R. B. KELMAN, Student, University of California, Berkeley, Calif.

R. R. KEMP, Student, McMaster University, Hamilton, Ontario, Canada

F. B. KEY, M.A. (Duke) Asst. Professor, University of Richmond, Va.

R. B. KILTIE, Student, Columbia University, New York, N. Y.

JACK KLUGERMAN, Student, Brooklyn College, N. Y.

KARL KOZARSKY, M.S. (Michigan) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

SANTOSH KUMAR, Student, Hindu College, Amritsar, India

C. H. LEE, Mathematical Statistician, Bureau of the Census, Washington, D. C.

ROY LEE, M.S. (Kansas S.T.C.) Professor of Chemistry, Barber-Scotia College, Concord, N. C.

R. A. LEIBLER, Ph.D. (Illinois) Mathematician, Sandia Corporation, Albuquerque, N. M.

STANISLAW LEJA, M.A. (Lwow) Dyer, Spinnerin Yarn Company, South Hackensack, N. J.

GILBERT LEMIEUX, C.E. (Ecole Poly. de Montreal) Asst. Principal Engineer, Highways Department of Quebec, Canada

C. J. LINDSAY, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

L. A. MAGNANTI, Student, Syracuse University, N. Y.

JOSEPH MAHLER, M.S. (Harvard) Project Engineer, Flight Research Laboratory, Dayton, Ohio

S. W. MALINOWSKI, Student, Rensselaer Polytechnic Institute, Troy, N. Y.

CARL MARKELLO, Student, University of Buffalo, N. Y.

ELIZABETH A. MARTZ, Student, University of Cincinnati, Ohio

R. D. MAYER, Student, Purdue University, Lafayette, Ind.

G. J. MAZZARA, Student, Hofstra College, Hempstead, N. Y.

J. H. MEANS, M.S. (Iowa) Asso. Professor, Huston-Tillotson College, Austin, Tex.

PINCHAS MENDELSON, Student, University of Pennsylvania, Philadelphia, Pa.

R. E. MONTGOMERY, M.S. (Illinois) Mathematical Research Analyst, Sandia Corporation, Albuquerque, N. M.

L. J. MONTZINGO, JR., M.A. (Buffalo) Instr., Roberts Wesleyan College, North Chili, N. Y.

B. J. MORSE, M.A. (Columbia) Grad. Student, Columbia University, New York, N. Y.

SHARON F. MURNICK, Student, Hofstra College, Hempstead, N. Y.

D. V. NEWTON, B.S. (Whitworth) Teaching Fellow, University of Washington, Seattle, Wash.

R. C. O'NEAL, B.S. in Ed. (Missouri) Teacher, Quincy High School, Quincy, Ill.

S. T. PAINE, B.A. (U.C.L.A.) Mathematician, Department of Meteorology, University of California at Los Angeles, Calif.

F. J. PALAS, Ed.M. (Oklahoma) Grad. Student, University of Oklahoma, Norman, Okla.

E. C. PALMER, B.S. Cashier, Palmer Electric Company, Seattle, Wash.

J. P. PIERCE, M.E.E. (Brooklyn Poly.) Asso. Professor, U. S. Naval Postgraduate School, Monterey, Calif.

D. H. Pilgrim, B.A. (Morningside) Grad. Student, State University of Iowa, Iowa City, Iowa

R. S. RAVEN, M.A. (Stanford) Senior Engineer, Westinghouse Electric Corporation, Friendship Airport, Baltimore, Md.

K. I. RAYMOND, Ed.M. (St. Lawrence) Com-

puter, Bell Aircraft Corporation, Niagara Falls, N. Y.

F. C. REED, B.S. (Redlands) Mathematician, Flight Determination Laboratory, White Sands Proving Ground, Las Cruces, N. M.

W. F. REYNOLDS, M.A. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

C. E. ROBERSON, Student, Carleton College, Northfield, Minn.

MAX ROSENBERG, M.S. (George Washington) Mathematician, Picatinny Arsenal, Dover, N. J.

E. M. SCHEUER, B.A. (Reed) Mathematician, Naval Ordnance Test Station, Pasadena, Calif.

E. M. SCHUSS, B.S. (St. John's) Corporal, U. S. Army, Camp Gordon, Ga.

J. A. SIMMONS, Student, University of California, Berkeley, Calif.

W. B. SIMMONS, JR., Student, Knox College, Galesburg, Ill.

D. A. SIPFLE, Student, Carleton College, Northfield, Minn.

SISTER M. DOMITILLA, Ed.M. (Marquette) Head, Mathematics and Physics Departments, Viterbo College, La Crosse, Wis.

SISTER MARY OF PERPETUAL HELP, M.A. (Catholic U.) Professor, Incarnate Word College, San Antonio, Tex.

J. H. SKELTON, M.A. (Missouri) Asst. Professor, Southwest Missouri State College, Springfield, Mo.

HAROLD SLATER, Student, Northeastern University, Boston, Mass.

F. B. SLOAT, M.A. (Arkansas) Asst. Professor, Kansas State College, Manhattan, Kan.

C. T. SMITH, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

P. L. SMITH, B.A. (Gustavus Adophus) Grad. Assistant, Kansas State College, Manhattan, Kan.

J. C. SOLLEY, Student, University of California, Berkeley, Calif.

D. R. SONDERGELD, Student, Bowling Green State University, Ohio

J. L. SPENCELEY, M.A. (Michigan) Instr., High School, Alpena, Michigan

W. L. STEINER, Student, Fordham University, New York, N. Y.

K. E. STOLL, Student, University of Kentucky, Lexington, Ky.

D. L. STURTZ, Student, Reed College, Portland, Ore.

NIYAZI TARIMER, Student, Massachusetts Institute of Technology, Cambridge, Mass.

G. T. THOMPSON, M.A. (Oregon) Grad. Assistant, Oregon State College, Corvallis, Ore.

M. W. TOLMAN, Student, University of South Dakota, Vermillion, S. D.

R. S. VARGA, M.A. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

J. F. WAMPLER, B.A. (Kansas) Instr., York College, Neb.

HELEN B. WARNER, B.S. (N.Y.S.C. for Teachers, Buffalo) Teacher, Board of Education, Buffalo, N. Y.

J. V. WHITTAKER, Student, University of California at Los Angeles, Calif.

R. E. WILD, M.S. (Iowa) Asst. Professor, University of Idaho, Moscow, Idaho

L. H. WILLIAMS, M.S. (Georgia) Mathematician, Redstone Arsenal, Huntsville, Ala.

J. K. WITTHAUS, Student, University of Oklahoma, Norman, Okla.

F. G. WITTMANN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

A. G. WOOTTON, M.A. (Columbia) Asst. Professor, Champlain College, Plattsburg, N. Y.

FUMIO YAGI, Ph.D. (M.I.T.) Asst. Professor, University of Washington, Seattle, Wash.

LOIS J. YOUNGER, Student, William Jewell College, Liberty, Mo.

THE TRAVELING LECTURER PROGRAM OF THE SOUTHWESTERN SECTION

Since its inception in 1937 the Southwestern Section of the Association has supported a traveling lecturer program for the promotion of interest in mathematics among the institutions of the section. Each year two lecturers are appointed by the executive committee of the Section, one to visit as many of the institutions in New Mexico as possible, the other to do the same in Arizona, and

to present a lecture on some mathematical subject intelligible to undergraduates. Except for the war years this program has been in operation with the enthusiastic support of a majority of colleges in the area.

In 1952 Dr. Morris Hendrickson of the University of New Mexico lectured in New Mexico on the subject *The Technique of Generalization as Applied to the Concept of Continuity*. Dr. D. L. Webb of the University of Arizona spoke in Arizona on *The Characteristics of a Mathematician*.

The program is supported financially by the institutions at which the lecturer appears. An amount of $25 is contributed usually by the participating colleges. However, the lecturer will visit institutions which cannot contribute that much or, indeed, will visit those which can provide no money if it does not involve too great inconvenience.

The members of the section feel that interest in mathematics has been promoted by the program and that it has aided materially in maintaining cordial relationships between the colleges and universities of the Southwestern Section.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-ninth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Northwestern Louisiana State College, Natchitoches, Louisiana, on February 15 and 16, 1952. Professor Arthur Ollivier, Chairman of the Section, presided at the Friday afternoon session. Professor C. G. Killen, Louisiana Vice-Chairman, presided at the Saturday morning session.

There were eighty-nine persons present including the following thirty-nine members of the Association:

T. A. Bickerstaff, Elsie T. Church, G. J. Corley, Margaret R. Davis, Virginia I. Felder, L. M. Garrison, M. E. Gillis, J. A. Hardin, Jessie M. Hoag, W. R. Hutcherson, L. H. Kanter, H. T. Karnes, C. G. Killen, S. R. Knox, Z. L. Loflin, A. C. Maddox, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, H. A. Palmer, P. K. Rees, T. L. Reynolds, F. A. Rickey, H. F. Schroeder, L. L. Scott, S. W. Shelton, P. K. Smith, W. H. Spragens, V. B. Temple, W. E. Timon, Jr., B. B. Townsend, G. J. Trammell, B. A. Tucker, B. O. Van Hook, W. M. Whyburn.

The following officers were elected for the coming year: Chairman, F. A. Rickey, Louisiana State University; Louisiana Vice-Chairman, Margaret M. LaSalle, Nicholls Junior College; Mississippi Vice-Chairman, T. L. Reynolds, Millsaps; Secretary-Treasurer, Z. L. Loflin, Southwestern Louisiana Institute.

The time and place of the next meeting were set as February 13–14, 1953 at Millsaps College, Jackson, Mississippi.

The invited speaker for the meeting was Professor W. M. Whyburn, University of North Carolina. His subject for an address at the Friday evening dinner held jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was *Efficient Use of Mathematically Trained Manpower*. At the Saturday morning session Professor Whyburn spoke on *Critical Solutions of Differential Systems*.

The following short papers were presented:

1. *Minkowski inequality for the mixed area*, by Miss Margaret M. LaSalle, Nicholls Junior College.

The area of a third convex curve traced out by the resultant of two vectors, each of which traces out a convex curve is given by

$$F = \frac{1}{2}\int (x_1 dx_2 - x_2 dx_1).$$

Substituting for the x's in terms of $p(\phi) = x_1 \cos\phi + x_2 \sin\phi$, the integrand becomes $p^2 - p'^2$, $p(\phi)$ of period π, limits of ϕ from $-\pi$ to π. Upon setting $p(\phi) = c_0 p_0 = c_1 p_1$ and rearranging terms according to subscripts there results

$$F = c_0^2 F_{00} + 2c_0 c_1 F_{01} + c_1^2 F_{11}$$

where F_{01} is the "mixed area." It is proved in this paper that the F's satisfy the Minkowski inequality

$$F_{01}^2 - F_{00}F_{11} \leqq 0.$$

2. *On the impossibility of an odd perfect number N with six different prime divisors*, by Professor T. L. Reynolds, Millsaps College.

Some lemmas and applications were given illustrating the proof of the following theorem: "On the Impossibility of an Odd Perfect Number N not Divisible by 5 with Six Different Prime Divisors"—the main theorem of the author's doctoral dissertation at the University of North Carolina in 1950.

3. *A new chapter in the life of Gauss*, by Professor G. Waldo Dunnington, Northwestern Louisiana State College, introduced by the Secretary.

This paper takes up the friendship of Gauss with numerous scientists and mathematicians, as well as individuals outside that classification, in foreign nations, particularly Great Britain and America. Particular attention is paid to B. A. Gould, the American astronomer, who was one of Gauss' pupils, to William C. Bond and his son George P. Bond, and to F. R. Hassler. Gauss' especial interest in America is discussed. His role in the affairs of the University of Göttingen and various attempts to lure him away from Göttingen form the final section of the paper.

4. *A certain cubic transformation*, by Elsie T. Church, Northwestern Louisiana State College.

This paper was published in this MONTHLY, vol. 59, p. 314.

5. *Invariant curves of order eight*, by Professor W. R. Hutcherson, University of Florida.

When invariant eight degree surfaces cut a certain fourth degree surface, the imperfect point P_3 is either an eight, seven, four, or one-tuple point. (W. R. Hutcherson, *Point non parfait et courbes invariables*, Bulletin de la Société Royale des Sciences de Liège No. 11 (1950).)

6. *A use of matrices in multivariant analysis*, by Professor S. R. Knox, Millsaps College.

A means of obtaining moments for the multi-variate normal distribution is shown, using methods suggested by Professor P. S. Dwyer of the University of Michigan.

7. *On the ultraspherical Christoffel numbers*, by Professor L. H. Kanter, Mississippi State College.

Let $w(x, \tau)$ be positive and continuous for $a<x<b$. Further, $w_\tau(x, \tau)$ shall exist and be continuous for $a<x<b$, and the integrals $\int_a^b x^\nu w_\tau(x, \tau)dx$, $\nu=0, 1, \cdots, 2n-1$ shall be uniformly convergent in every closed interval $\tau' \leqq \tau \leqq \tau''$ of the open interval (τ_1, τ_2). Then differentiating the Gauss-Jacobi mechanical quadrature formula with respect to τ, we have $\int_a^b w_\tau(x, \tau)\rho(x)dx = \sum_{\nu=1}^{n}\lambda_\nu(\tau)\rho'(x_\nu)x_\nu'(\tau)+\sum_{\nu=1}^{n}\lambda_\nu'(\tau)\rho(x_\nu)$. Here $\rho(x)$ is an arbitrary polynomial of degree $2n-1$ and $\lambda_\nu(\tau)$ are the Christoffel numbers. If for $\rho(x)$ we choose the well-known Hermite polynomial of the first kind corresponding to the ultraspherical polynomial $P_n(x, \tau)$, it may be shown that $\lambda_\nu(\tau)$ is a decreasing function of τ for $-1<\tau\leqq 0$.

8. *N-section curves*, by Professor V. B. Temple, Louisiana College.

Circles with centers at O_1 and O_2 and radii a and r are tangent at A, internally or externally. They rotate around their centers with no slipping at A. H_1 and H_2 are points on the circles such that $AH_1=AH_2$. Let θ be the angle AO_1H_1 and α the angle AO_2H_2. Then, if $a=nr$, $n\theta=|\alpha|$, $(n>0)$. Now draw lines through H_1 and H_2 parallel to O_1O_2 to meet lines through H_1 and H_2 at right angles respectively in P_1 and P_2. The locus of each of these points is an *N-section curve*.

If O_1 is the origin and O_1O_2 the positive end of the x-axis, the parametric equations are:

$$P_1: \quad x = a\cos\theta, \quad y = \frac{a}{n}\sin n\theta, \qquad P_2: \quad x = \frac{a}{n}(n \pm \cos n\theta), \quad y = a\sin\theta,$$

where the upper and lower signs in x of P_2 are respectively for the conditions of external and internal tangency. Charts of these curves were shown for $n=2, 3, 4, 5$, and 6.

9. *Wrong method, right answer*, by Professor F. A. Rickey, Louisiana State University.

A student in elementary algebra "solved" the simultaneous pair

$$y = x + 2$$
$$(x+1)^2 + (y-5)^2 = 16$$

by replacing the second of these equations by the pair $x+1+y-5=\pm 4$. The four points obtained are actually the intersection points of the given pair.

In this paper an analysis of the conditions under which such incorrect procedure yields the correct answers was given. Other instances in which incorrect procedures result in the correct answers were discussed. A study of the Diophantine equation for the cancellation

$$\frac{1\beta}{\beta 4} = \frac{1}{4}$$

leads to the interesting cancellation

$$\frac{270}{756} = \frac{20}{56}.$$

10. *Finite metabelian groups and lines of a projective 5-space*, by Professor L. L. Scott, University of Mississippi.

A finite metabelian group G is a group whose commutator subgroup K is contained in the central C. The metabelian groups considered are such that: $K=C$; all elements are of order p; each group is generated by six elements and is not the direct product of an abelian group and a

metabelian group with less than six generators. One-to-one correspondences were set up between certain abelian subgroups of G and the points of a projective 5-space S_5 and between the cyclic subgroups of order p of C and the points of a projective 14-space Σ_{14}. It was shown that an element of C is a commutator of G if and only if its corresponding point in Σ_{14} lies on the eight-dimensional manifold V determined as the locus of the Plücker coordinates of the lines of S_5.

F. A. RICKEY, *Secretary*

THE MARCH MEETING OF THE KANSAS SECTION

The thirty-seventh annual meeting of the Kansas Section of the Mathematical Association of America was held at Bethany College, Lindsborg, Kansas on March 29, 1952. Professor Anna Marm, Chairman of the Section, presided at both the morning and afternoon sessions.

One hundred seventy persons were present including the following forty-three members of the Association:

P. H. Arnold, Wealthy Babcock, Florence L. Black, Jeneva J. Brewer, J. E. Faulkner, D. H. Firl, W. C. Foreman, Abraham Franck, Albert Furman, W. H. Garrett, F. C. German, Laura Z. Greene, (Mrs.) Sabrina Hecht, D. M. Houser, W. C. Janes, L. E. Laird, K. E. Lake, Violet H. Larney, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, Agnes E. Nibarger, V. D. Nyhoff, S. T. Parker, O. J. Peterson, C. B. Read, L. M. Reagan, Sister M. Helen, Sister Jeanette, Sister M. Nicholas, R. G. Sanger, W. R. Scott, G. W. Smith, R. G. Smith, E. C. Stopher, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, Frances E. Walsh, A. M. Wedel, A. E. White, Ferna E. Wrestler, P. M. Young.

At the business session the following officers were elected: Chairman, P. M. Young, Kansas State College; Vice-Chairman, W. C. Foreman, Baker University; Secretary-Treasurer, Laura Z. Greene, Washburn University.

The program consisted of the following papers:

1. *Mathematics in the secondary school, 1952*, by Professor J. R. Mayor, University of Wisconsin.

In this paper an analysis was given of factors affecting the place and nature of mathematics instruction in the secondary schools and of the trends in secondary school mathematics. Special reference was made to the current shortage of scientific personnel, advantages of co-operation between college and secondary school teachers, inservice programs for teachers, curriculum plans which tend to de-emphasize mathematics, enrollment statistics, and course modifications at the secondary school level. The conclusion was drawn that there was evidence to support an optimistic point of view.

2. *Adequate opportunity for the gifted high school student*, by Mr. A. S. Richert, Wichita High School East, Kansas, introduced by Professor Anna Marm.

A democracy must provide educational opportunity commensurate with ability. About 10 per cent of the six million high school children have an I.Q. better than 120. These are "the gifted." Certain barriers hamper the gifted such as: inflexible curriculum and teaching methods; resentment of teachers toward the gifted; lack of time, energy, equipment for special help. Education for the gifted is improved by: improved methods of identification; improved counseling; a chance to work in chosen field before graduation; financial assistance even in high school; acceleration, homogeneous grouping, more electives, enrichment. The New York High School of Science is a good example of adequate opportunity. The plan of the Wisconsin University Correspondence School illustrates what can be done for smaller high schools.

3. *Opportunity for the gifted student in junior college*, by Miss Henrietta Courtright, Arkansas City Junior College, Kansas, introduced by Professor Anna Marm.

The best brains of society are required to meet the need for consecrated leadership in American civilization. Junior colleges can help by identifying gifted students, providing opportunities to develop their talents by acceleration, sectioning, enrichment and elective courses, guiding them to grasp their opportunities, and developing their sense of social responsibility. Scholarships should be attainable by financially handicapped gifted students. Teachers with imagination, enthusiasm and ability to transmit live, meaningful knowledge can help the gifted realize the responsibility of their talents and harmonize personality with intellectual development. Gifted students must be educated to the height of their powers for the realization of these ideals.

4. *Opportunities for the gifted student in the university and the college*, by Professor R. G. Sanger, Kansas State College.

In this paper the opportunities available in small colleges and large universities for students with special mathematical abilities were considered. The amount of individual attention given to a student in each type of school, the variety of offerings, physical facilities, and the stimulation gained by contact with similar interests were discussed.

5. *Certain functions defined by series*, by Professor Ferna E. Wrestler, University of Wichita.

This paper used infinite series to define two functions $C(x)$ and $S(x)$ and to develop several of their properties. It was then shown that these are the familiar properties of the functions cos x and sin x which are usually developed in terms of angles.

6. *Point sets and number theory*, by Professor E. C. Stopher, Fort Hays Kansas State College.

Certain analogies between point sets and numbers were discussed. Equations involving point sets, their sums and their products were presented, together with corresponding equations involving numbers, their lowest common multiples and their greatest common divisors.

7. *Remarks on Tchebycheff polynomials*, by Mr. P. H. Arnold, Kansas State College.

The speaker discussed the problem proposed by Tchebycheff in 1854: Given a polynomial in x of the form, $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, to determine that one which in the interval, $-1<x<1$, possesses the smallest maximum absolute value. He outlined a general method of deriving these polynomials and presented a few special properties of them.

8. *Certain congruent figures*, by Professor J. D. Haggard, Kansas State Teachers College, introduced by the Secretary.

A number of examples and theorems concerning congruent figures in euclidean space were reviewed. The examples and theorems seem to violate one's intuition and were suggested as an antidote for the attitude often expressed by students that a certain theorem is "obvious" and therefore does not require a proof.

LAURA Z. GREENE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at Agnes Scott College and Georgia Institute of

Technology, Decatur, Georgia, on March 21–22, 1952. Professors B. G. Clark, Chairman of the Section, H. A. Robinson and J. W. Lasley, Jr., presided over the general sessions; and Professors B. G. Clark, H. K. Fulmer, Vice-Chairman of the Section, and W. L. Williams presided over subsections.

There were about two hundred fifty present including the following one hundred twenty-six members of the Association:

Edith W. Ainsworth, S. A. Anderson, P. L. Armstrong, A. H. Bailey, J. C. Barnes, D. F. Barrow, W. S. Beckwith, M. G. Boyce, Evelyn Boyd, G. M. Brown, J. W. Brown, R. K. Butz, H. E. Campbell, L. Virginia Carlton, Ella F. Casey, R. S. Christian, B. G. Clark, C. E. Clark, A. B. Coble, A. C. Cohen, Jr., R. M. Conkling, C. L. Cope, F. E. Cothran, R. W. Cowan, J. C. Currie, J. J. Dennis, Annie L. Dicks, V. E. Dietrich, N. E. Dodson, Patrick DuVal, L. A. Dye, J. C. Eaves, David Ellis, Trevor Evans, Floyd Field, Tomlinson Fort, R. W. Frankel, H. K. Fulmer, Barbara A. Futral, L. L. Garner, J. R. Garrett, Leslie J. Gaylord, V. D. Gokhale, D. B. Goodner, S. T. Gormsen, H. C. Griffith, S. W. Hahn, E. H. Hadlock, E. A. Hedberg, Melvin Henriksen, A. T. Hind, C. H. Holton, C. W. Hook, J. H. Hornback, J. V. Howell, H. B. Hoyle, Jr., C. W. Huff, G. B. Huff, S. L. Hull, L. P. Hutchinson, S. L. Jamison, Ayrlene M. Jones, F. W. Kokomoor, J. W. Lasley, Jr., J. W. LaGrone, C. G. Latimer, R. J. Levit, F. A. Lewis, Lee Lorch, R. A. Lytle, S. W. McInnis, Nathaniel Macon, J. D. Mancill, J. M. Marr, C. F. Martin, W. A. Martin, G. W. Medlin, Benjamin Evans Mitchell, J. C. Morelock, T. F. Mulcrone, W. V. Neisius, Sara L. Nelson, T. A. Newton, J. D. Novak, W. V. Parker, Erna H. Pearson, Lillian G. Perkins, C. G. Phipps, W. W. Rankin, Ellen F. Rasor, G. E. Reves, A. J. Robinson, H. A. Robinson, J. W. Sawyer, Carol S. Scott, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, C. Eucebia Shuler, C. B. Smith, E. L. Stanley, L. W. Stark, F. W. Stallard, A. L. Starrett, J. M. Thomas, H. E. Taylor, S. L. Thompson, J. C. Thurman, H. S. Thurston, Lila P. Walker, D. T. Walker, J. G. Wall, Susie L. Ward, W. W. Weber, M. C. Wicht, L. H. Williams, W. L. Williams, Ernest Williams, L. E. Williams, R. L. Wilson, Herbert Wolf, G. N. Wollan, R. S. Wollan, F. L. Wren, J. W. Young, B. K. Youse.

At the business session the following officers were elected: Chairman, H. K. Fulmer, Georgia Institute of Technology; Vice-Chairman, W. V. Parker, Alabama Polytechnic Institute; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its next meeting in March, 1953, at Alabama Polytechnic Institute, and its 1954 meeting at University of South Carolina.

The program consisted of the following papers:

1. *Coordination of mathematics with industry*, by Professor W. W. Rankin, Duke University.

As chairman, Professor Rankin gave a preliminary report of a committee of twelve mathematicians from industry and six from teaching on the coordination of mathematics with industry.

2. *Historical outlines helpful in teaching mathematics*, by Professor G. E. Reves, The Citadel.

Samples of historical outlines for arithmetic, algebra and geometry were presented. A discussion was given of the usefulness of such outlines in helping the student understand both the development of these subjects and their subject matter. The importance of this approach in motivating the student in the teaching of college mathematics was emphasized.

3. *The merits and content of a freshman mathematics course*, by Professor F. L.

Wren, George Peabody College for Teachers.

Is our traditional program the best program possible in mathematics for college freshmen? There is a need for a more basic course for all people capable of taking a college education. There are certain fundamental postulates which constitute the foundation upon which such a course might be constructed.

4. *Mathematics in the research program of the Ordnance Corps*, by Dr. J. J. Gergen, Chief pro tem, Mathematics Division, Ordnance Research.

For many years a substantial amount of basic research has been carried out at Ordnance research and development installations. In 1950 the Ordnance Corps incorporated into its general program a program of sponsored research under the Office of Ordnance Research. This office is interested in basic research in essentially all mathematical fields. Of special interest are those topics in mathematics which have more immediate bearing on Ordnance related subjects such as ballistics, plasticity, and heat conduction. Proposals by qualified investigators on problems of particular interest to the investigators are encouraged.

5. *Unified mathematics since 1936*, by Professor B. G. Clark, Vanderbilt University.

Ninety-five schools were included in the survey with at least one from each state except one. In 1936, 20 of the schools offered a unified course; in 1941, 31; then perhaps due to the war the number decreased to 18; and for the last three years 21. It is estimated that not more than 7% of all freshmen now are in these courses. The fact that a unified course requires more experienced instructors, and the difficulty experienced when a student transfers has contributed to the hesitancy of most schools to adopt a unified approach to mathematics.

6. *One-sided maxima and minima of functions of two or more variables*, by Dr. J. D. Mancill, University of Alabama and Redstone Arsenal.

Dr. Mancill considered a real single-valued function $z=f(x, y)$ of two real variables, which is continuous on a closed region R, bounded by an ordinary curve C. The extreme values of $f(x, y)$ may be at points interior to R or at points on its boundary C. Necessary and sufficient conditions were determined in order that $f(x, y)$ have an extreme value at (x_0, y_0) on C relative to the region R of admissible points (one-sided extremes). These results were extended to functions of more than two variables.

7. *A representation of an analytic normal operator*, by Professor S. L. Jamison, Florida State University.

The following theorem was proved: If $N(z)$ is analytic and normal for $|z|<\rho$, then there is a self-adjoint operator A and a function $F(z, \lambda)$ such that $N(z)=F(z, A)$ where $F(z, \lambda)$ is analytic in z for each fixed λ and is A-integrable with respect to λ for $|z|<\rho$.

8. *Magic circles*, by Professor S. W. McInnis, University of Florida.

This paper classifies magic circles into simple, compound, and perfect. The development of the last two was demonstrated by the formula $S_r=c+(a+b)x/2$, S_r being the sum of the number along any radius, a the lower bound, b the upper bound, x the number of rings, and c an arbitrary constant.

9. *On the Euler method of summability for double series*, by Professor G. N. Wollen, Memphis State College.

In this paper the Euler method of summability was defined for double series and a few of its

properties discussed.

10. *The exponent of a product mod M*, by Mr. R. S. Wollen, University of Georgia.

The author discussed the known results on a special case of the exponent of a product modulo M and showed they might be extended to include other special cases.

11. *Applications of the stress and strain tensors to orthotropic materials*, by Professor C. B. Smith, University of Florida.

Since in the use of orthotropic material the applied forces may not act always in the direction of the axes perpendicular to the planes of elastic symmetry, the expression of the relations between stress and strain when referred to any orthogonal axes are sometimes needed. By a simple use of the stress and strain tensors, the author readily obtained the desired relations and gave some applications.

12. *The ratio equation of a conic*, by Professor W. L. Williams, University of South Carolina.

By the process of elimination among certain triangular area ratios, Professor Williams showed how the equation of a general conic could be written in terms of these ratios. As an immediate consequence, a theorem on a quadrilateral inscribed in a circle was extended to hold for any pentagon.

13. *On the element of* -1 *dimensions in projective geometry*, by Professor Patrick Du Val, University of Georgia.

A plea was made for the formal introduction in the presentation of projective geometry of a unique element of -1 dimensions through which all elements of the geometry are held to pass and is accordingly the intersection of two elements which in the usual sense do not intersect. This is required by the principle of duality; it is needed to complete the validity of the formula for the dimensions of the join and intersections of two given elements; and if the geometry is developed in terms of homogeneous coordinates the void corresponds to the set $(0, \cdots, 0)$ and if elements are regarded as aggregates of points it is the empty aggregate.

14. *Topological group isomorphisms*, by Mr. W. L. Strother, University of Alabama Center, Gadsden, introduced by the Secretary.

A new proof and a generalization were made of the first and second isomorphism theorems for topological groups as stated, for example, in Pontrjagin's *Topological Groups*.

15. *A note on the rotation of axes*, by Professor R. L. Wilson, University of Tennessee.

Professor Wilson reviewed familiar facts concerning the rotation of axes in analytic geometry as applied to conic sections, and suggested some labor saving devices in connection with the reduction of an equation to the standard form.

16. *Arithmetic—algebra—geometry—each an aid in the study of the other*, by Professor W. W. Rankin, Duke University.

The author exhibited several simple instruments for making the "any" concept more acceptable to students, also some instruments which would give the idea of " parameter" and "function" a more friendly welcome.

17. *Imbedding of normed Abelian groups*, by Mr. R. M. Conkling and Pro-

fessor David Ellis, University of Florida.

Any normed Abelian group which satisfied $\|ma\| = |m|\|a\|$, where m is any integer and a is any element of the group, may be isometrically isomorphically imbedded in a Banach space.

18. *Cross-associativity and essential similarity in double-sfields*, by Professor David Ellis, University of Florida.

In any double-sfield $R(+, *, \sharp)$, cross-associativity $[a\sharp(b*c) = (a\sharp b)*c;\ \forall a, b, c \epsilon R]$ and essential similarity $[\exists k \epsilon R \ni a*b = a\sharp k\sharp b;\ \forall a, b \epsilon R]$ are equivalent and each implies isomorphism of the two mutiplications $a\sharp b$ and $a*b$.

19. *Sylvester's technique for solving the equation $X^2+PX+Q=0$ in binary matrices*, by Professor H. S. Thurston and Miss Mary K. Alexander, University of Alabama, introduced by the Secretary.

Sylvester obtained a cubic equation from which the characteristic equations of all solutions of $X^2+PX+Q=0$ can be obtained. In this paper the special case in which P is a linear function of Q is discussed in detail.

20. *Reduction of a matrix to rational cononical form*, by Professor Lila P. Walker, Woman's College of University of North Carolina.

Comparisons of various methods are given for finding the matrix P such that $P^{-1}AP$ gives the rational cononical form for A. Certain observations are made concerning the minimum functions of the column vectors of the matrix P.

21. *Criteria for normal matrices*, by Professor Benjamin Evans Mitchell, Alabama Polytechnic Institute.

Fourteen necessary and sufficient conditions for a matrix to be normal were listed, including some new ones. Proofs for the latter were given.

22. *The derivatives of Lagrangian coefficients in terms of Vandermonde determinants*, by Professor C. L. Seebeck, Jr., University of Alabama.

Let $V = |(x-x_i)^j|$, where i and j range from 0 to n, be a Vandermonde determinant; V_i^j the cofactor of the element in the i-th row and j-th column; $L_i^j(x) =$ the j-th derivative of $L_i(x) = \prod_{i \neq k}(x-x_k)/(x_i-x_k)$. Then $L_i^j(x) = j!\,V_i^j/V$.

23. *A system with ternary operation*, by Professor J. C. Currie, Georgia Institute of Technology.

An algebra with one ternary operation was defined axiomatically, the axioms being chosen in analogy to the axioms for a group. Some additional axioms were proposed and a number of simple theorems developed.

24. *Properly primitive indefinite ternary quadratic genera of more than one class*, by Professor E. H. Hadlock, University of Florida.

Let certain restrictions be put on the invariants of a form and let the characters have assigned values. Then a form exists with certain conditions attached to its coefficients. By applying Siegel's method to the form, a method suggested by B. W. Jones, genera containing at least three classes are shown to exist.

25. *A new type of planar birational transformation*, by Emeritus Professor A. B. Coble, University of Illinois (visiting professor, University of North

Carolina).

The conventional method of transforming a curve with compound singularities into one with ordinary nodes is first to reduce the compound singularities to ordinary singularities (multiple points with distinct tangents) by a succession of quadratic transformations each of which may introduce additional ordinary singularities; and then by birational transformations to replace the ordinary singularities by nodes. In this paper a (1,3) transformation set up by a net of conics with a single base point (F-point), and thus only one P-curve, is applied to obtain the first stage in the reduction of a compound singularity at the expense of introducing nothing but ordinary nodes.

26. *An application of Laplace's transformation to a class of linear partial differential equations of the third order*, by Professor R. W. Cowan, University of Florida.

A change of independent variables is made so that the form of the differential equation is simplified. The form of the reduced equation depends upon the roots of a certain cubic equation. Several cases arise depending on the nature of these roots. If the coefficients of the reduced equation satisfy certain conditions, a general solution may be found by solving a Lagrange linear equation.

27. *A note on compound frequency distributions*, by Professor A. C. Cohen, Jr., University of Georgia.

This paper concerns the conditions under which two normal frequency distributions with different means may combine to produce a compound distribution that is unimodal rather than bimodal.

28. *On a class of polynomials*, by Professor F. A. Lewis, University of Alabama.

The purpose of this paper is to present some properties of the polynomials P_n, defined by

$$\frac{d^n}{dx^n}(1-x^2)^{-1/2} = (1-x^2)^{-(2n+1)/2}P_n.$$

29. *Lattices of integers*, by Professor J. M. Thomas, Duke University.

The points of the plane whose Cartesian coordinates are integers form a lattice. A necessary and sufficient condition that a lattice for an oblique system be also a lattice for a rectangular system is found.

30. *A finite formula for the Bernoulli numbers*, by Professor E. B. Shanks, Vanderbilt University.

By use of complete induction, it is shown that the determinant A_j given by the author in an article "Iterated sums of powers of the binomial coefficients," this MONTHLY, Vol. 58, No. 6, p. 405 has the value

$$A_j^k = \sum_{r=0}^{j-1} (-1)^r \binom{ik+1}{r} \binom{i+j-1-r}{i}^k.$$

If we now set $i=1$ in the latter formula and $i=p=1$ in formula (6) of the same article, then determine the coefficient of n, we obtain as a finite formula for the k-th Bernoulli number

$$B_k = \sum_{j=1}^{2k} (-1)^{j+1} A_j^{2k} (2k+1)^{-1} \binom{2k}{j-1}^{-1}.$$

31. *Expected increase in earning due to college training*, by Professor S. L. Thompson, Alabama Polytechnic Institute.

If one considers college training as only a financial investment that may be regained by increased earnings, he can express his potential annual earning E_x as a function of his age x, his number of years of college training, retirement age, probabilities of living each of the years until retirement, college fees paid, interest rate on investment, relative values of investments at different times and the average annual wages received by persons without college training. The fact that each year of college training represents an investment of a year of potential earning, gives the function E_x somewhat of a recursion relationship.

32. *Integral rational functions and irrational roots of integral rational equations in freshman algebra*, by Professor H. B. Hoyle, Jr., Queens College.

A method was presented for investigating algebraic functions and for finding irrational roots of an algebraic equation utilizing such parts of calculus and theory of equations as may be brought within the easy grasp of college freshmen.

33. *Remarks on radical equations*, by Professor G. B. Huff, University of Georgia.

This paper was published in this MONTHLY, vol. 59, p. 320.

34. *A geometric discussion of certain radical equations*, by Professor D. F. Barrow, University of Georgia.

This paper was published in this MONTHLY, vol. 59, p. 320.

35. *Partial differentiation in elementary calculus*, by Professor L. A. Dye, The Citadel.

A closely integrated treatment of the topics usually included in the theory of partial differentiation is developed without the use of infinitesimals or the Mean Value Theorem.

36. *A note on exact linear differential equations*, by Professor W. V. Neisius, Georgia Institute of Technology.

The principle of exact linear differential equations was discussed and a simple method of solving was shown which enables most students to test for exactness by inspection and, if exact, to write down the answer immediately.

37. *Matrices and polynomials*, by Professor W. V. Parker, Alabama Polytechnic Institute.

In this paper the "companion" matrix for a monic polynomial is considered. A proof of the Hamilton-Cayley Theorem is given for this matrix. Several other theorems concerning polynomials in this matrix are given.

38. *A heat flow problem*, by Mr. O. R. Ainsworth, University of Alabama, introduced by the Secretary.

Mr. Ainsworth presented a corrected solution to a heat flow problem which he felt was improperly reported in the January *Mathematical Reviews*.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-second annual meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, on Saturday, March 8, 1952. Mr. William H. Glenn, Chairman of the Section, presided.

The attendance was one hundred fifteen, including the following eighty members of the Association:

O. W. Albert, Florence R. Anderson, T. M. Apostol, L. A. Aroian, H. M. Bacon, Leon Bankhoff, R. S. Barton, May M. Beenken, Clifford Bell, L. T. Black, Gertrude Blanch, H. F. Bohnenblust, Frances L. Campbell, L. M. Coffin, E. P. Coleman, P. H. Daus, Robert Davies, R. P. Dilworth, D. C. Duncan, H. P. Edmundson, W. V. Gamzon, F. W. Gibson, W. H. Glenn, M. L. Goldwater, C. J. A. Halberg, H. J. Hamilton, V. C. Harris, Robert Hartranft, A. F. Herbst, R. B. Herrera, M. R. Hestenes, P. G. Hoel, J. M. Howell, D. H. Hyers, P. B. Johnson, G. R. Kaelin, Rosella Kanarik, L. C. Lay, D. H. Lehmer, J. W. Lindsay, Sophia L. McDonald, Fred Marer, F. R. Morris, W. A. Nielsen, P. M. Niersbach, L. J. Paige, D. B. Perry, D. J. Peterson, W. T. Puckett, C. A. Pursel, H. R. Pyle, E. C. Rex, J. M. Robb, E. B. Roessler, J. M. Sandy, I. J. Schoenberg, G. E. F. Sherwood, Sister Rose Gertrude, Samuel Skolnik, James C. Smith, R. H. Sorgenfrey, I. S. Sokolnikoff, Robert Steinberg, J. D. Swift, T. E. Sydnor, Anna L. Taylor, D. L. Thomsen, Elmer Tolsted, C. W. Trigg, S. E. Urner, F. A. Valentine, D. D. Wall, Maria Weber, J. G. Wendel, R. L. White, A. L. Whiteman, Lois B. Whitman, B. R. Wicker, F. M. Yanari, E. M. Zaustinsky.

At the business meeting the following officers were elected for the next academic year: Chairman, C. W. Trigg, Los Angeles City College; Vice-Chairman, A. E. Taylor, University of California at Los Angeles; Secretary-Treasurer, P. H. Daus, University of California at Los Angeles; Program Committee: L. Clark Lay (Chairman), John Muir College; T. M. Apostol, California Institute of Technology; A. L. Whiteman, University of Southern California.

The next meeting was scheduled for March 14, 1953, at Los Angeles City College.

The following program was presented:

1. *The uniform step method for magic squares*, by Professor T. M. Apostol, California Institute of Technology.

This paper has appeared in the Proceedings of the American Mathematical Society, vol. 2 (1951), pp. 557–565, under the title *On Magic Squares Constructed by the Uniform Step Method.* The paper was written jointly with H. S. Zuckerman of the University of Washington.

2. A panel discussion: *What can collegiate mathematics contribute to the education and training of high school teachers?* Participants: Mr. Dale Carpenter, Los Angeles City Schools, introduced by Professor P. H. Daus; Dr. D. C. Duncan, East Los Angeles Junior College; Professor H. J. Hamilton, Pomona College.

Mr. Carpenter suggested the following procedures to college mathematics teachers: development of a specific plan for helping train mathematics teachers; a definite way of cooperating with the education and teacher training departments; cooperative planning directed toward the reduction of the number of hours now required for a teaching major; cooperation with secondary teachers in planning institutes, conferences, workshops; visitation of secondary school mathematics classes; cooperation with high school teachers in revising and improving mathematics content and

method at the secondary level; formulation and practice of good instructional procedures for college mathematics classes; familiarity with the applications of mathematics in consumer mathematics, business education, industrial arts, chemistry, physics; provision of courses pertaining to research in mathematics curricula, and mathematics instruction, methods of teaching arithmetic, general mathematics, algebra, geometry and trigonometry, arithmetic content, and basic relationships of arithmetic, algebra, geometry and trigonometry.

To enable and to encourage high school mathematics teachers to widen their mathematical culture, Dr. Duncan suggested that the local universities offer degrees of D.M. (Doctor of Mathematics) and D.M.S. (Doctor of Mathematical Sciences), upon the completion of 75 units of mathematics, or of mathematics and allied subjects, accumulated with grades of A or B within a period of 15 years. This suggestion was referred to the Education Committee of the Southern and Northern California Sections for study.

Professor Hamilton suggested that the Association would do well to define the teaching *minor* in mathematics as well as the teaching major; also, to recommend that potential high school teachers take courses in the history of mathematics, the teaching of mathematics and (perhaps in high school itself) a physical science, descriptive geometry, and public speaking. He urged college teachers to employ good teaching techniques.

3. *Machine development and research in pure mathematics*, by Professor D. H. Lehmer, Institute for Numerical Analysis, National Bureau of Standards.

The present day development of computing machinery was discussed from a purely mathematical point of view. The advantages and disadvantages of certain characteristic features of machines as applied to problems in pure mathematics were treated in some detail. The possible future influence of computing machinery upon research mathematics and mathematical points of view was discussed briefly.

4. *On bell-shaped curves*, by Professor I. J. Schoenberg, Institute for Numerical Analysis, National Bureau of Standards.

Let $f(x)$ ($\not\equiv 0$) be defined for all real x, be infinitely often differentiable and such that $f^{(n)}(x) \to 0$, as $x \to \pm\infty$, for $n = 0, 1, \cdots$. Let N_n denote the number of changes of sign of $f^{(n)}(x)$. Rolle's theorem readily implies that $N_n \geqq n$ for all values of n. The curve $y = f(x)$, and also the function $f(x)$, is called *bell-shaped* if $N_n = n$ for all values of n. The bell-shaped function $f(x)$ is said to be *bell-shaped in the interval* (a, b) provided $f(x) = 0$ outside (a, b). While many examples of functions are known which are bell-shaped in $(-\infty, \infty)$, $[\exp(-x^2), (1+x^2)^{-1}]$, or in $(0, +\infty)$, $[f(x) = x^{-1}\exp(-x^{-1})$ if $x > 0, f(x) = 0$ if $x < 0]$, I. I. Hirschman has proved the following theorem: *There exist no functions $f(x)$ which are bell-shaped in a finite interval (a, b)*. Hirschman's elegant and elementary proof (Proceedings American Mathematical Society, vol. 1 (1950), pp. 63–65) was presented.

5. *Experiences teaching mathematics in Japan*, by Professor Frances L. Campbell, George Pepperdine College.

The following observations were made from 1948 to 1950 while the speaker was teaching at Ibaraki Christian College, a small college in rural Japan.

Scientific studies are, with a few exceptions, about 20 years behind the present developments in the States. The students of mathematics seem to be well prepared in the elementary courses, with emphasis on computation. Formulas of courses also emphasize computational ability and rote learning, influenced perhaps by Japan's earliest contribution to mathematics, a well developed system of computation called *Wasan*. The Japanese in general have the ability but not the initiative to be creative. There is a national organization for mathematicians and teachers of mathematics. Most school libraries have few recent mathematical publications in English. The entire educational program is in a state of transition under the impact of the occupation. The old system

was very rigid with very strict entrance examinations and a set course of study, allowing no transfer of credit between courses or between universities. The new system is an attempt to integrate modern trends of the West with the good points of the old system. Accreditation to teach in Japanese colleges requires an advanced degree but only those subjects may be taught in which the prospective teacher has had articles published, even though the field may differ from that of the degree.

6. *Semi-groups and Laplace transforms*, by Professor R. S. Phillips, University of Southern California, introduced by Professor D. H. Hyers.

Let X be a Banach space and let $T(s)$ be a one-parameter family of linear bounded transformations on X to itself. If $T(s_1+s_2)=T(s_1)T(s_2)$, then these operators form a semi-group of transformations. It is further assumed that $\lim_{s\to 0}T(s)f=f$ for each $f\epsilon X$. If further f belongs to the domain $D(A)$ of the infinitesimal generator A, then $\lim_{\Delta\to 0}(T(s+\Delta)f-T(s)f)\Delta^{-1}$ exists and is equal to $AT(s)f$. The theory of semi-groups is applicable to partial differential equations in the following way. Consider for example $u_s=u_{xx}$, $-\infty<x<\infty$, $s\geqq 0$ and $u(0, x)=f(x)$, where say $f\epsilon L_1(-\infty, \infty)$. Then there exists a semi-group of operators such that $u(s, x)=T(s)f$. To establish the existence of this type of solution it is sufficient to show, according to a theorem due to Hille and Yosida, that $\|R(\lambda; A)\|\leqq(\lambda-\omega)^{-1}$ for $\lambda>\omega$. Here $R(\lambda; A)$ is the inverse of $(\lambda I-A)$. In general this criterion is not too difficult to verify.

7. *Geometric background of some theorems of Hardy, Littlewood, and Polya*, by Dr. Seymour Sherman, Lockheed Aircraft Corporation, introduced by Professor M. R. Hestenes.

The Muirhead theorem in Hardy, Littlewood and Polya's *Inequalities* concerning real numbers and its analogue for general vector space is related to the duality theory of convex polyhedral cones in finite dimensional spaces.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-ninth annual meeting of the Iowa Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Iowa Academy of Science at Coe College, Cedar Rapids, Iowa, on April 18–19, 1952. Rev. L. E. Ernsdorff, Chairman of the Section, presided.

A total of forty-three persons registered their attendance including the following twenty-two members of the Association:

E. R. Bowersox, F. A. Brandner, E. L. Canfield, J. O. Chellevold, N. B. Conkwright, W. M. Davis, L. E. Ernsdorff, A. M. Feyerherm, B. E. Gillam, Cornelius Gouwens, J. J. L. Hinrichsen, Don Kirkham, C. H. Lindahl, R. B. McClenon, J. V. McKelvey, E. E. Moots, Fred Robertson, F. M. Stein, E. C. Stopher, H. P. Thielman, Roscoe Wood, and C. C. Wylie.

Officers elected at the meeting are: Chairman, L. A. Knowler, State University of Iowa; Vice-Chairman, J. O. Chellevold, Wartburg College; Secretary-Treasurer, Fred Robertson, Iowa State College. The next meeting is to be held at Cornell College, Mt. Vernon, Iowa, on April 17–18, 1953.

Professor H. P. Thielman was granted the award for the most meritorious paper. His paper is in competition with similar papers from the other sections of the Iowa Academy of Science for a fifty dollar prize.

The following papers were presented:

1. *On the zeros of* $J_n(x_p)Y_m(kx_p)-J_m(kx_p)Y_n(x_p)$, by Mr. Don Kirkham, Iowa State College.

Explicit expressions for kx_p as a function of x_p are derived for the indicated equation, in which J_n and Y_m are the first and second kind of Bessel functions of order n and m respectively, k is a parameter and p gives the order of the root x_p. Similar expressions are derived when the indicated equation is modified so that the inner two terms or the outer two terms or all four terms are primed, that is, differentiated with respect to the argument. The derived expressions are asymptotic and yield better values of kx when $|m|$ is small and x_p and p are large. The derivations utilize methods of Stokes and MacMahon (See Watson's *Bessel Functions*, p. 505). The results differ from MacMahon's expressions in that values $x_p=0$ are not excluded. In fact, when $m=n=0$ and $p=1$, kx_p is given to an accuracy of better than 1 part in 1500 by the formula when $x=0$. For cases when the inner and outer terms are primed, some earlier sets of roots are found than heretofore mentioned in the literature. For these roots, which may be termed zeroth order roots, the analytical expressions derived above are not valid for $x_p=0$. For calculation of the zeroth order roots graphical methods are presented.

2. *Laguerre series in the complex domain: The representation problem*, by Professor Nelson Yeardley, Iowa State College, introduced by the Secretary.

Let $\{L_n^\alpha(x)\}(n=0, 1, 2, \cdots; \alpha>-1)$ represent a sequence of Laguerre polynomials of order α and degree n. To the function of a complex variable $f(z)$, $(z=x+iy)$, let correspond the general Fourier-Laguerre series of $f(z)$:

$$f(z) \sim \sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z) \qquad (\alpha > -1)$$

where a_n^α represent the Fourier-Laguerre coefficients. If $d_\alpha = -\limsup (2n^{1/2})^{-1} \log |a_n^\alpha|$, then for $d_\alpha>0$ the general Fourier-Laguerre series of $f(z)$ converges absolutely inside and diverges outside the parabola of convergence $p(d_\alpha)$: $y^2=4d_\alpha^2(x+d_\alpha^2)$.

The author proves the following theorem: *If the general Fourier-Laguerre series of $f(z)$ converges at a point $z_0=x_0+iy_0(y_0\neq 0)$ then to every $b_0(0<b_\alpha<d_\alpha)$ there corresponds a positive number $B(\alpha, b)$ such that for all points z (with $|z|$ large) inside the parabola of convergence $p(d_\alpha)$ of the general Fourier-Laguerre series of $f(z)$*

$$\left|\sum_{n=0}^{\infty} a_n^\alpha L_n^\alpha(z)\right|^2 < B(\alpha, b)\frac{\sqrt{2}}{16}\frac{\Gamma(\alpha+1)}{q^{3/4}}\frac{x^{1/4}}{|z|^{\alpha+1/2}}e^{z-2\sqrt{qz}}\left(1+\frac{a_1}{x^{1/2}}+\frac{a_2}{x^{2/2}}+\cdots\right)$$

where

$$q=c_\alpha^2-\tfrac{1}{2}(r-x); \qquad r=|z|; \qquad c_\alpha=\tfrac{1}{2}(b_\alpha+d_\alpha).$$

3. *Conjugate points of singular quadratic functionals for N dependent variables*, by Professor J. O. Chellevold, Wartburg College.

Morse and Leighton (*Singular quadratic functionals*, Trans. Amer. Math. Soc., vol. 40, 1936 pp. 252–286) gave a systematic approach to the problem of minimizing a singular quadratic functional for one dependent variable. They defined the first conjugate point of the singular point $x=0$ and determined the analogue of the Jacobi condition. In this paper the definition of the conjugate point is extended to n dependent variables and resulting theorems are proved. The analogue of the Jacobi condition as a necessary condition for a minimum limit for a class of curves defined as C-admissible is also extended to n dependent variables.

4. *Probability of success in business mathematics*, by Professor E. L. Canfield,

Drake University.

Predicting achievement in a business mathematics course at Drake University involves use of a test designed to be given at the beginning of a semester. Test scores for 118 students were used to develop a discriminant function under the assumptions of biserial correlation, which function, in the case of one variable, may be expressed in the form: bx. The coefficient b may be found by solving the equation $D=b\sum x^2$, where D is the difference between the means of the test scores in the unsuccessful and successful groups of business mathematics students, and the right-hand member of the equation is the same as in regression analysis using deviation values from the general mean.

By using an appropriate transformation of the function, a value in sigma units can be computed for any particular student, given his predictive test scores, that yields a probability upon consulting a table of the normal curve. A tabulation of such probabilities may be used in advising a student of his chances for success in business mathematics. Such probabilities might be developed using several variables, as in multiple regression.

5. *The application of overlapping Pfaffians to utility theory*, by Professor C. H. Lindahl, Iowa State College.

A sub-set of the integrability conditions is imposed upon overlapping sub-Pfaffians of length greater than two, and the coefficients restricted sufficiently to make feasible an analysis of common class two. It is proved that these conditions imply exactness when the coefficients are polynomials of the first and second degree or general functions under a specific restriction.

Properties of the above Pfaffians are applied to utility theory in economics. The distinction between integrability and non-integrability is considered as a distinction between rational and irrational behavior on the part of the consumer. Make the general hypotheses: (a) a consumer will be rational in an economic transaction if the number of variables involved is sufficiently, but not trivially, restricted, and (b) a consumer cannot only order the variables of the above transaction and therefore group them in harmony with such an order but there exists an ordering such that he can deal rationally within the groups provided the size of the groups is less than four. Then the following conclusions are drawn: (a) the consumer's utility Pfaffian in general is not only integrable, in the large, but is exact, and (b) the implication is that the behavior of the consumer, in the large, is rational in general.

6. *A note on autocorrelation for an oscillating series*, by Professor F. A. Brandner, Iowa State College.

For oscillating series it is difficult to find what periods exist, or whether any definite periodicity prevails. A number of methods have been used to gain information for that type of a time series curve. Among those methods are two which have proved of greater use than any of the others, autoregression and serial correlation. However, the two methods are used separately and have not been interrelated.

In this paper a method of finding autocorrelation by use of the autoregression equation is given. The results are summarized as to what was accomplished, with its advantages and disadvantages.

7. *Equivalent functions of strategies in the theory of games*, by Mr. Harlan Mills, Iowa State College, introduced by the Secretary.

By use of the restrictions on the strategy vectors, x, y, in the bilinear payoff form, xAy, we obtain a relation which holds for every pair of strategies, x, y,

$$xAy = uAv + (x - u)B(y - v),$$

where u, v are functions of the coefficients of A, and B is a somewhat arbitrary linear combination of matrices derived from the coefficients of A. If A is square and non-singular, the condition

for existence of the form is that the sum of the cofactors of A be non-zero.

When $u^i \geqq 0$, all i, and $v^j \geqq 0$, all j, u and v are optimal and uAv is the value of A. B has sufficient freedom to include any arbitrary pair of strategies in its own optimal sets, and have arbitrary value. B also includes A. When a Taylor expansion exists with all first derivatives zero, it is included in the form above.

8. *Distribution of Kronecker products*, by Mr. A. M. Feyerherm, Iowa State College.

Let $K = B \times C$ where K, B and C are square matrices with real elements and $(\times)$ denotes the Kronecker product of B and C. The elements of a particular $K_{n \times n}$ may be considered as the coordinates of a point in n^2 dimensional space. The distribution of the points representing $K_{n \times n}$'s is considered in this paper. Insight into the density of such points is obtained by finding the K or K's which minimize the tr $(A-K)(A-K)'$ where A is a given matrix which is not an exact Kronecker product. Results are derived for a number of specific forms of A and K.

9. *Pathological functions*, by Professor H. P. Thielman, Iowa State College.

Let f be a function defined on X onto Y. Let P be a point property of the function f. The function f is said to be peculiar or pathological with respect to the property P if there exists a partition of X into two subsets X_1 and X_2 each of which is everywhere dense in X, and the property P holds at every point of X_1 and fails to hold at every point of X_2. Examples were given of functions which were pathological with respect to the properties of continuity, neighborliness and differentiability. For the definitions of neighborliness of a function see an article by W. W. Bledge; (Proc. Amer. Math. Soc., vol. 3, 1952, pp. 114–115). Cliquishness of a function was defined as follows: Let the domain of definition of a function f be a neighborhood space X with neighborhoods N_x. Let the range of the function f be a metric space with metric P. Then the function f is said to be cliquish at a point x_0 if for every neighborhood N_{x_0} of x_0 and for every positive number ϵ there exists a neighborhood N_x contained in N_{x_0} (but not necessarily containing x_0) such that for every two elements x_1 and x_2 of N_x it is true that $P(f(x_1), f(x_2)) < \epsilon$. It was proved that there exists no function which is pathological with respect to cliquishness. The concept of neighborliness of a function was extended to functions whose domains of definition and ranges are general neighborhood spaces. Among the results on cliquish functions were the following: The set of points where a pointwise cliquish function is cliquish is nowhere dense. The limit of a sequence of cliquish functions may have no points of cliquishness, and hence may be totally discontinuous. Every cliquish function is at most pointwise discontinuous.

10. *Regularization of systems of differential equations*, by Professor J. J. Hinrichsen, Iowa State College.

In this paper the transformation method applied so successfully by Levi Civita (*Acta Mathematica*, Vol. XXX (1906)) and G. D. Birkhoff (*Rendiconti del Circolo Matematico di Palermo*, Vol. XXXIX (1914)) in regularizing the equations of motion of the restricted problem of three bodies, is used to regularize the equations of motion of certain other dynamical problems.

11. *The path and orbit of the Oklahoma detonating meteor of November 7, 1951*, by Professor C. C. Wylie, State University of Iowa.

Throughout the week of November 18–25, 1951, the author, assisted most of the time by Professor B. S. Whitney of the University of Oklahoma, interviewed observers of this meteor, having them stand exactly where they were when the meteor was seen and measuring the angles of the path as recalled. Some additional measures were made by O. E. Monnig of Fort Worth, Texas.

From these measures and from stop watch timings of "reenactions of the scene," the path through the atmosphere and the orbit in which the meteor had been traveling about the sun have been computed.

12. *An experiment in teaching algebra and trigonometry*, by Professor Fred Robertson, Iowa State College.

The author discussed an experiment in teaching in which an attempt was made to teach the usual material in college algebra and trigonometry in one quarter instead of the customary two. He described the methods used and gave some of the results in statistical form. The results indicate the older method is the preferred one.

13. *Notes on the mean value theorem*, by Reverend L. E. Ernsdorff, Loras College.

Very often students who study the mean value theorem are puzzled about how the usual function $F(x)$ is derived. Geometric ways of deriving the function which is the difference between the ordinate and chord on the curve were explained.

14. *An aid to teaching the metric system*, by Professor Boyd Henry, Parsons College, introduced by the Secretary.

The author described the method of constructing a motion picture accompanied by sound which he uses to teach the metric system of weights and measures. The cost is not prohibitive since the total outlay for material was $12.50. The film was run and the section saw it from the viewpoint of the students in class.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, on April 19, 1952. Dr. J. A. Ward, Chairman of the Section, presided at the morning and afternoon sessions.

There were seventy-six persons in attendance, including the following thirty-six members of the Association:

H. H. Berry, M. C. Brown, R. C. Brown, H. W. Burnette, Esther A. Compton, H. H. Downing, R. I. Fields, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, E. J. Musch, R. S. Park, Sallie E. Pence, Mary Pettus, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, Sister M. Rosalin, D. C. Rose, F. E. Ross, W. C. Royster, D. E. South, Guy Stevenson, R. P. Tapscott, J. T. Vallandingham, J. A. Ward, R. H. Wilson, H. A. Wright, R. C. Yates, W. M. Zaring.

Officers elected at the meeting were: Chairman, Professor W. L. Moore, University of Louisville; Secretary, Professor J. A. Ward, University of Kentucky.

The annual meeting in 1953 is to be held at the University of Louisville.

The following papers were given:

1. *The development of operational calculus*, by Professor Tadeusz Leser, University of Kentucky.

The author described briefly the Heavyside operational calculus and mentioned the earlier operational methods. The Heavyside expansion formula for $1/\phi(p)$ operating on the unit function $H(t)$ was justified through the Laplace transformation. Attention was drawn to the fact that the Heavyside operator p with rational exponents positive or negative is not justified as yet. The out-

line of the present state of the operational calculus, based on the Laplace transformation, was also presented.

2. *A note on the teaching of plane analytic geometry*, by Professor Aughtum S. Howard, Kentucky Wesleyan College.

As an aid in curve sketching in plane analytics, it is suggested that $y=f(x)$ and $y=1/f(x)$ be consistently paired for study when ever feasible. By drawing the lines $y=1$ and $y=-1$, a visual demonstration of the relationship between the zeros of $f(x)$ and the vertical asymptotes of $y=1/f(x)$ is possible. In polar coordinates, consideration of $r=1$ along with a pairing of $r=f(\theta)$ and $r=1/f(\theta)$ shows graphically the relationship between the zeros of $f(\theta)$ and the oblique asymptotes of $r=1/f(\theta)$. This approach also makes possible an elementary introduction to the study of inversion.

3. *Ceva's point for a problem in geometry*, by Professor W. J. Robinson, Centre College.

The following problem was considered. For a given triangle ABC, find point O such that $OA:OB:OC=r_1:r_2:r_3$. The solution of this problem depends on the location of Ceva's point P. An analytic derivation of the locus of P yields four conics. Each of these passes through A, B, and C. No two intersect otherwise. They are tangent by twos at A, B, and C and the tangent lines are the angle bisectors of triangle ABC.

4. *Drawings illustrating the problem of Apollonius*, by Professor H. A. Wright, Transylvania College.

After a brief description of methods of constructing circles tangent to three given circles, a series of large scale drawings was exhibited. The drawings consisted of twelve relative positions of the given circles with all possible (zero to eight) tangent circles constructed. The actual constructions were done by a student, Mr. D. C. Rose, as a project in college geometry.

5. *The use of calculus for solving heat problems*, by Professor R. H. Wilson, University of Louisville.

Integration concepts essential to chemical thermodynamics were exemplified as most cogent reasons for the study of calculus by chemistry majors. In the relation $\Delta H=\int_{T_1}^{T_2}C_p dT$, H represents the heat content, C_p the heat capacity constant pressure, which is a function of T, the absolute temperature of the given mass. Such an integral may add understanding even to the homely process of cooking, where, since the top limit on T is usually a definite boiling point, the degree of cooking is a simple function of the more easily measured element, time. In extreme-temperature chemical reactions, the top limit on T may be the most inaccessible quantity in this relation, which thus becomes an integral equation. Such was the case for the hydrogen-fluorine flame, recently investigated by the speaker as consultant of the Research Institute of Temple University. For this extreme temperature, 4300° K, the problem is complicated by the necessity of considering, simultaneously with the integral equation, the relations giving equilibrium constants for the three reversed reactions, as functions of the unknown partial pressures of the three substances thus formed. A trial-and-error solution indicated the world's highest flame temperature, which was later confirmed experimentally only by direct comparison with the sun.

6. *Giant brains*, by Professor D. W. Pugsley, Berea College.

The author became interested in so called "giant brains" while studying at the Watson Scientific Computing Laboratory, New York City, when on sabbatical leave. In this paper he listed some of the calculators that are in operation or under construction, with brief statistics about them. The several types of problems solvable by such machines were described. Then followed a discussion of several systems of numbers commonly employed for computations in giant

brains, namely, the binary, the bi-quinary, and the octal number systems. The author concluded with a statement of the prospects for the future, both as to machines and personnel, and with a plea to other teachers to encourage promising students to investigate this field.

7. *The straight line*, by Professor A. W. Goodman, University of Kentucky.

Using the Zermelo well-ordering theorem, discontinuous solutions of the functional equation $f(x+y)=f(x)+f(y)$ can be constructed in the manner of G. Hamel, *Math. Ann.* vol. 60, 1905, pp. 459–462.

8. *The place of mathematics in the life adjustment program*, by Sister M. Matilda, Loretta Junior College, introduced by the Secretary.

From the very nature of the life-adjustment program no definitive place can be assigned to any particular subject, but as the study of mathematics in some form or other prepares for intelligent and happy daily living for all, it should rank high in the curriculum. Gauging space, size, measure, distance come constantly into daily life as also do buying, selling, computing, accounting. In business or in leisure, in urban or in rural living, at any age, all need mathematics; hence its importance. Steam, internal combustion, hydroelectric and atomic power projects call for experts with upper-bracket scientific education and technical training based on highly specialized mathematical foundations, but this concerns the few. What of the others? Besides its practical usefulness mathematics exerts a disciplinary influence on the mind, training to habits of thought that carry over into other areas, and, if properly taught even to those on the lower levels, it can afford pleasure and entertainment, and can lead to appreciation of higher things and provide wholesome and worthy use of leisure time.

9. *Measure theory*, by Mr. F. E. Ross of Lybrand, Ross Brothers and Montgomery, Louisville.

After stating the problem of measure, the more significant theories which endeavor and purport to find its solution are discussed in historical sequence. Thus, there are defined the theories due to Hankel-Stolz-Harnack, Cantor, Peano-Jordan, Borel, Lebesgue, Caratheodory and Haar, and theorems are proven regarding these measures and measure-systems. The properties stressed in this presentation give the extent of the class of sets measurable under these various theories and attempt to clarify the relationships among the measures discussed.

10. *The probability of x successes in n trials*, by Mr. R. E. Wheeler, University of Kentucky, introduced by the Secretary.

This paper is concerned with finding the probability of x successes in n trials, $P(n, x)$, where the probability of success on a single trial varies depending on the number of the trial and on the number of previous successes. A probability function is algebraically formulated and $P(n, x)$ is obtained in terms of this function. The Bernoulli probability is considered to illustrate the general procedure.

11. *Models, methods and the stimulation of interest in the teaching of college mathematics*, by Dr. R. C. Yates, United States Military Academy.

A nucleus of ideas intended to stimulate and maintain interest in various phases of high school and college mathematics was presented. Items discussed in connection with algebra, geometry, analytics, calculus, mechanics, and optics included: (1) the crossed parallelogram linkage; (2) the ladder and rolling circles; (3) roses and the hypocycloids; (4) maximum-minimum solutions by elementary methods; (5) caustics and their reflecting curves; (6) l'Hospital's rule; (7) asymptotes and tangents at the origin; (8) sums of powers of integers; (9) normal and tangential acceleration; (10) Euler's exponential formula.

A variety of models and diagrams were used to emphasize these ideas and to give point to the general theme. It was recommended that a small laboratory space be established and supplied with inexpensive materials for classroom activities to foster and encourage the spirit of "to have and to hold."

12. *A study of caustic curves*, by Mr. A. G. McGlasson, Eastern Kentucky State College.

A caustic curve is the envelope of light rays emitted from a radiant point source after reflection or refraction by a given curve which may be considered as the boundary separating two optical media. The idea of caustics originated with Huygens and Tschirnhausen about 1678. It was remarked that the actual determination of the caustic of a general plane curve often presents great difficulty but that the calculation can be performed for simpler curves. Gergonne has shown that each caustic is the evolute of an algebraic curve and it was noted that it is often more convenient to express the caustic in this manner. Some of the more interesting caustics were discussed and illustrated.

13. *Failures caused by fear*, by Sister M. Fides, St. Catharine Junior College, introduced by the Secretary.

Many failures of mathematics students are caused, not by lack of ability, but by fear of the subject or some phase of it. Considered as a disease, a term, mathemaphobia, is coined for this attitude. Causes of mathemaphobia are: (1) unpleasant childhood experiences connected with the subject, (2) absences from school, causing a gap in successive stages of development on which satisfactory performance of later stages depend, (3) faulty teaching methods in which stress is placed on practice rather than on understanding of techniques, (4) failure to make the effort to master certain phases of the subject, and (5) practice of teachers whereby skills are shown without any explanation of the application and thought necessary to acquire such skills. Some suggestions for diagnosis and treatment of mathemaphobia are drawn from a long teaching experience. These are, briefly: (1) getting the student to look at his own case objectively and work toward a cure, (2) helping a student to bridge the gap after absences, (3) emphasis on teaching *why* rather than *how*, (4) recognition of those certain areas which cause fear with special patience and understanding of students in these difficulties, and (5) ability on the part of teacher to show that skills are not acquired ready-made but are purchased by thought and application.

14. *On the product representation of sin* πx, by Professor V. F. Cowling, University of Kentucky.

In this paper the product representation of sin πx is developed without the aid of the theory of functions of a complex variable.

J. A. WARD, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Virginia Military Institute, Lexington, Virginia, on April 26, 1952. Professor E. R. Sleight, Chairman of the Section, presided at the morning and afternoon sessions.

There were sixty-five persons in attendance, including the following thirty-four members of the Association:

W. E. Byrne, H. H. Campaigne, G. R. Clements, Georgie T. Davis, L. R. Ford, Herta T. Freitag, C. H. Frick, F. S. Goepper, Michael Goldberg, E. S. Grable, M. Gweneth Humphreys,

J. E. Ikenberry, Mary A. Lee, Florence P. Lewis, R. H. Kasriel, F. B. Key, Ella C. Marth, F. H. McGar, Jr., Carol V. McCamman, C. G. Maple, T. W. Moore, E. K. Paxton, K. S. Purdie, L. V. Robinson, W. G. Saunders, Veryl G. Schult, E. R. Sleight, R. E. Smith, R. F. Steward, Mildred E. Taylor, Marian M. Torrey, R. E. Walters, P. M. Whitman, G. T. Whyburn.

During the business meeting the following officers were elected: Chairman, Professor Marian M. Torrey, Goucher College; Vice-Chairman, Professor M. Gweneth Humphreys, Randolph-Macon Woman's College; Vice-Chairman, Professor Mildred E. Taylor, Mary Baldwin College; Secretary, Professor C. H. Frick, Mary Washington College.

It was announced that the next meeting of the Section will be held at Howard University, Washington, D. C., on December 6, 1952. The Section voted to amend the By-Laws of the Section by replacing the words "additional members" by "Vice-Chairmen" on line two in section one of Article III. It was further voted that the Committee on proposals for future activities of the Section be thanked for their efforts and that authority be given to proceed with steps that might be necessary.

The following papers were presented at the meeting:

1. *The partial differential equations of visco-elastic materials*, by Professor R. E. Smith, College of William and Mary.

The partial differential equations of visco-elastic media and the solution of certain boundary value problems were considered. The materials considered were isotropic. Both compressible and incompressible materials were considered giving Alfrey's incompressible results as a special case. Changes of state were isothermal and only small strains were considered. The stress-strain relations used were

$$Ps_{ik} = 2Qe_{ik}, \qquad \sigma = K\epsilon, \text{ where } K \text{ is a constant.}$$

The s and e are the deviatoric stress and strain, σ and ϵ are the mean stress and strain, and P and Q are differential operators in t.

2. *Some novel sequences*, by Professor E. S. Grable, University of Richmond.

The method of iteration, commonly used to approximate roots of an equation, was shown to be a source of unusual sequences when applied to suitably chosen equations with known roots. In particular the equation $X=\sqrt{2^x}$ leads to a sequence, involving repeated extraction of roots, whose limit is 2, 4, or ∞ depending upon the initial value of X used. The equation $aX^2-bX=0$ in the form $X=\sqrt{(a/b)X}$ leads to another sequence involving the repeated extraction of roots. The equation $X=\int_0^x t^{-1/2}dt$ leads to a sequence involving repeated integration. Functions defined in terms of the limits of these sequences were considered.

3. *Solution of a most general form of the Clairaut equation by differential operator methods*, by Dr. L. V. Robinson, Aberdeen Proving Ground.

Defining

$$\lambda \sim P_1\frac{\partial}{\partial x_1} + P_2\frac{\partial}{\partial x_2} + \cdots + P_n\frac{\partial}{\partial x_n},$$

where P_k, $(k=1, 2, \cdots, n)$ are functions of x_k, it was shown how equations of the type

$$u = (\lambda^{-1}\cdot 0)\lambda u + F(\lambda u)$$

can be solved by operator methods.

4. *A proposed course for college freshmen*, by Professor W. E. Byrne, Virginia Military Institute.

The ideas of rational numbers, enumerable and non-enumerable sets, arithmetic progressions, geometric progressions, two-dimensional coordinate systems, vectors, definition of the trigonometric functions in terms of the general angle, complex numbers, and partial fractions in forms suitable for presentation to freshmen were discussed.

5. *Proposals for future activities of the section*, by Professor S. B. Jackson, University of Maryland, Professor Ella C. Marth, Wilson Teachers College, Professor Florence M. Mears, George Washington University, and Miss Veryl G. Schult, District of Columbia Public Schools; presented by Miss Schult.

The desirability of sponsoring contests for high school and college students, preparing a bulletin for high school students showing the mathematics requirements for various college courses offered in the area, expanding our relations with industry, having group meetings on the teaching of college mathematics, and participation of graduate students in section programs was discussed.

6. *Some remarkable theorems about areas*, by Dr. L. R. Ford, Emeritus Professor of Mathematics, Illinois Institute of Technology.

This was the invited lecture. A number of general theorems were presented. Application of some of these to the figures studied in a first course in calculus was made.

C. H. FRICK, *Secretary*

THE APRIL MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section of the Mathematical Association of America was held on April 12, 1952 at the University of Michigan in Ann Arbor in conjunction with the annual meetings of the Michigan Academy of Science, Arts, and Letters. Professor Norman Anning, Chairman of the Section, presided at both morning and afternoon meetings as well as at the business meeting held immediately following the luncheon at the Michigan Union.

A total of seventy-four persons attended the meetings including the following fifty-four members of the Association:

M. W. Al-Dhahir, H. W. Alexander, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, J. H. Bell, J. W. Bradshaw, G. U. Brauer, D. M. Brown, C. H. Butler, R. V. Churchill, Nathaniel Coburn, S. D. Conte, A. H. Copeland, J. W. Coy, D. E. Deal, M. L. DeMoss, P. S. Dwyer, C. M. Erikson, J. S. Frame, E. W. Goings, Betty Grossman, G. W. Grotts, V. G. Grove, A. E. Hallerberg, H. H. Hannon, Louise F. Hanson, G. E. Hay, Fritz Herzog, E. E. Ingalls, P. S. Jones, Wilfred Kaplan, A. E. Lampen, H. D. Larsen, R. D. Lowe, L. E. Mehlenbacher, D. M. Mesner, H. W. Nace, A. L. Nelson, R. S. Pate, Mary H. Payne, J. E. Powell, G. Y. Rainich, E. D. Rainville, P. H. Raker, C. C. Richtmeyer, E. H. Rothe, Arthur Saastad, J. L. Spenceley, E. M. Steinbach, H. E. Stelson, B. M. Stewart, D. T. Teodoro.

At the business meeting the nominating committee, consisting of Professors Fritz Herzog and A. L. Nelson, reported the names of Professor H. D. Larsen for Chairman and Professor P. S. Jones for Secretary-Treasurer for the year 1952–53. They were elected unanimously.

The 1953 meeting of the Section was set for April 11 in Kalamazoo at Western Michigan College of Education.

The following papers were presented at the morning and afternoon meetings:

1. *Complex numbers as an example of the development of mathematical ideas*, by Professor P. S. Jones, University of Michigan.

The role of a course in the history of mathematics in displaying the growth of and changes in fundamental concepts from their inception until the present was illustrated by tracing the development of complex numbers from Graeco-Arabic times through W. R. Hamilton's formulation of complex numbers as pairs or "couples." The interest and value to be derived by referring to the formulations of the early authors were pointed out by means of slides showing selected pages from original works.

2. *The simple geometric proof of the minimax theorem*, by Professor A. H. Copeland, Sr., and Dr. Howard Raiffa, University of Michigan.

The minimax theorem is a consequence of the fact that in Euclidean n-space two convex sets can be separated by a hyperplane provided they have at most boundary points in common and at least one of the sets contains interior points. The authors gave a simple geometrical proof of such a separation. The proof is valid for Hilbert space. The results are applied to the study of convex (polyhedral) cones and to the theory of linear programming without recourse to the equivalence of the sum and intersection definitions of convex polyhedral cones. For completeness sake an alternate proof of this equivalence was given.

3. *The parabolic correlation coefficient*, by Professor W. D. Baten, Michigan State College.

This paper showed that a non-linear correlation coefficient pertaining to a polynomial predicting equation, for normally distributed independent variables, can be expressed as a function of the square of the averages of the derivatives of the predicting equations for the given values.

4. *Discontinuities in compressible fluid flow*, by Professor Nathaniel Coburn, University of Michigan.

By following a method developed by R. K. Luneberg for studying electromagnetic waves (*Asymptotic Development of Steady State Electromagnetic Fields*, N.Y.U., Mathematics Research Group, Research Report No. EM 14, July 1949), a unified theory of discontinuities (shock and contact manifolds, characteristic manifolds) for the case of compressible fluids is obtained. Essentially, the method depends upon writing the equations of motion, continuity, and energy in divergence free form in a space-time manifold. By applying Stokes' theorem, one may express the previous equations as integrals over hypersurfaces of space-time. These integrals are assumed to be the fundamental equations of the theory. In regions where the density, velocity, *etc.* are continuous with continuous derivatives, these equations reduce to the original equations of motion, continuity, and energy. If the density, velocity, *etc.* are discontinuous over certain hypersurfaces, then the equations furnish relations for the various discontinuities along contact and shock manifolds. By use of a differentiation process, discontinuities of the derivatives of density, *etc.* along characteristic manifolds can be studied.

5. *Teaching a blind student in a regular college course in algebra*, by Mr. E. M. Steinbach, University of Detroit.

The problems involved in teaching a completely blind student in a regular class in intermediate algebra were discussed, some of the solutions arrived at by the student and instructor working together were outlined, and some sources of materials helpful in such a situation were mentioned.

6. *The Waring problem in the dissection of polygons*, by Professor B. M. Stewart, Michigan State College, and Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

Given n congruent polygons of shape P the problem is to dissect each polygon in the same way into K parts so that the total set of Kn parts may be composed, without reflection, exactly to fill a large polygon, directly similar to the original smaller polygons. The authors succeeded in finding a value for K *independent* of n; in fact if one of the given polygons can be divided into e triangles, it was shown that $K(n, P) \leqq 7e$. For a parallelogram M, $K(n, M) \leqq 6$. For a square S, $K(n, S) \leqq 4$ and for many values of n they are able to produce interesting three-part and two-part dissections of the square.

7. *Finitely normal numbers*, by Mr. H. A. Hanson, Michigan State College, introduced by the Chairman.

The solution of problem 4385 in the MONTHLY, [1951, 573], in which a number is constructed to the base 2, in which every sequence of n digits occurs exactly once, suggests the extension of such a construction to an arbitrary base $B \geqq 2$. The construction consists in writing n digits $B-1$, followed at each position thereafter by the smallest digit which does not repeat a sequence of n digits previously occurring. A finitely normal number of order n is then defined as a decimal, expressed in some base B, in which every sequence of n digits occurs with a limiting frequency of $1/B^n$. If the first B^n digits of the number constructed as above are taken as the period of a repeating decimal, the resulting decimal is finitely normal of order n, but of no order greater than n.

8. *Modifications of Fourier integrals*, by Professor R. V. Churchill, University of Michigan.

Let $z(x, y)$ denote the static transverse displacements in a stretched membrane covering the quadrant $x>0$, $y>0$. The edge $y=0$ has a prescribed displacement $F(x)$. The edge $x=0$ is attached to the center line of a stretched metal strap that is rigid in the x-direction but bends like a stretched string in the yz-plane, producing a boundary condition of the type $z_{xx}(0, y) = kz_x(0, y)$. Separation of variables in this boundary value problem leads to a solution provided that $F(x)$ can be represented by a modification of the Fourier cosine integral formula in which the kernel function $\cos \lambda x$ is replaced by the function $\cos(\lambda x+\beta)$, where $\tan \beta = \lambda/k$. A short derivation of that formula was presented. The corresponding modified Fourier transforms and their applications were noted, as well as other trigonometric modifications of Fourier integrals.

9. *The solution of the bilateral matrix equation*, by Professor J. H. Bell, Michigan State College.

This paper deals with the bilateral matrix equation $\sum_{m=0}^{s} \sum_{i=0}^{m} X^{m-i} A_{m-i,i} X^i = 0$, where all matrices involved are square of order n and have elements in a field $\mathfrak{F}$. A brief summary of the work done in special cases of this problem was given. Necessary conditions were developed which suggested a method of solution of the general equation. Let $Y = (Y_{ij})$ where $Y_{ij} = X^{i+j-2}$, and $B = (B_{ij})$ where $B_{ij} = A_{i-1,\, j-1}$ $(i = 0, 1, 2, \cdots, s; j = 0, 1, 2, \cdots, s; A_{kj} = 0, j > k)$. If B is nonsingular and X is a solution of the equation then Y must be such that $Y = ZB^{-1}$ where $Z^s = 0$. If B is singular, Y must be such that $Y = ZQP$ where

$$PBQ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

I and Z_{11} are $r \times r$ submatrices; P, Q are non-singular, $Z_{11}^s = 0$ and $Z_{21}Z_{11} = Z_{11}Z_{12} = Z_{21}Z_{12} = 0$.

10. *A reducible case*, by Professor N. H. Anning, University of Michigan.

It was suggested as a classroom device that, after a cubic equation has been prepared for

solution, it be identified with

$$x^3 - 3m^2x - 2n^3 = 0.$$

The solution of this "by Cardan" is tidy. Moreover if there are three real roots the students can put $n^3 = m^3 \cos 3A$ and by an easy application of De Moivre's Theorem go ahead to the solutions: $x_1 = 2m \cos A$, $x_2 = 2m \cos (A + 120°)$, $x_3 = 2m \cos (A + 240°)$.

P. S. JONES, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-sixth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 19, 1952. Professor R. F. Rinehart, Chairman of the Section, presided at the morning and afternoon sessions.

Ninety-seven persons registered in attendance, including the following seventy-seven members of the Association:

J. E. Adney, Jr., A. G. Anderson, W. E. Anderson, Grace M. Bareis, I. A. Barnett, William Beck, Theodore Bennett, G. M. Bloom, Foster Brooks, Dorothy I. Carpenter, A. B. Carson, V. B. Caris, W. G. Clark, Florentina M. Clinton, W. F. Cornell, H. K. Crowder, J. E. Darraugh, C. H. Denbow, B. B. Dressler, O. L. Dustheimer, E. R. Epperson, D. T. Finkbeiner, II, M. P. Fobes, H. G. Harp, Frances Harshbarger, R. G. Helsel, Carl Holtom, R. Y. Iwanchuk, E. D. Jenkins, Margaret E. Jones, Chosaburo Kato, K. D. Kelly, L. C. Knight, D. M. Krabill, D. H. Kraft, Nathan Lazar, F. C. Leone, H. D. Lipsich, C. P. Louthan, L. L. Lowenstein, W. J. McCallion, S. W McCuskey, H. R. Mathias, Margaret E. Mauch, L. H. Miller, Max Morris, O. M. Nikodym, S. R. Orr, P. M. Pepper, H. S. Pollard, W. O. Portmann, Tibor Rado, R. F. Reeves, P. R. Rider, L. G. Riggs, Louis Ross, H. J. Ryser, Samuel Selby, Sister M. Constantia, Ruth B. Smyth, E. T. Stapleford, Andrew Sterrett, C. W. Topp, E. P. Vance, W. R. VanVoorhis, E. H. Wang, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, E. H. Wohler, Alberta Wolfe, G. F. Woodson, Jr., C. H. Yeaton, Marie M. Yeaton, B. J. Yozwiak, A. D. Ziebur.

The following officers were elected for the coming year: Chairman, E. J. Mickle, The Ohio State University; Secretary-Treasurer, Foster Brooks, Kent State University; third member of the Executive Committee, H. J. Ryser, The Ohio State University; Program Committee: W. R. Transue, Kenyon College, Chairman; L. L. Lowenstein, Kent State University; Marion Wetzel, Denison University.

A special committee on methods for promoting interest in mathematics among secondary school students, Professor F. B. Wiley, Ashland College, chairman, reported on methods in use at present, and recommended, first, that a project for this purpose be launched on a nation-wide basis as a joint effort of the Mathematical Association of America and the National Council of Teachers of Mathematics; and, second, that such a joint project be planned at once for Ohio.

The following papers were presented:

1. *Functions of matrices*, by Professor R. F. Rinehart, Case Institute of Technology.

A system of first order linear homogeneous differential equations with constant coefficients,

when written as a single matric differential equation, gives rise to the search for a matric solution analogous to that of the similar appearing single scalar differential equation, namely the matric exponential function. A discussion of the power series representation of the matric exponential function was followed by a discussion of the general matric functions defined by power series, including conditions for convergence. Since every convergent power series in a square matrix Z can be written as a polynomial in Z, it was shown that this polynomial was necessarily that given by the Sylvester-Bucheim definition of a function of a square matrix, and hence that the latter definition is more inclusive than the power series definition.

The definitions of matric function given by Schwerdtfeger, Fantappie, Jordan, Giorgi, and Cipolla were discussed. The equivalences of the Schwerdtfeger and Sylvester-Bucheim definitions, and of the Fantappie and Jordan definitions were noted. It was remarked that the combinatory criteria laid down by Fantappie are satisfied by all of the definitions except, under certain conditions, that of Cipolla. These criteria insure that a polynomial identity among scalar functions will be preserved by their matric analogues, according to any of the above definitions (with the exception noted). Thus, for example, $\sin^2 Z+\cos^2 Z=1$, $e^{iZ}=\cos Z+i \sin Z$, hold for square matrices Z.

2. *Rigid membranes*, by Professor T. Rado, The Ohio State University.

It has been known for over thirty years that a flexible and inextensible membrane of *convex* shape is infinitesimally rigid if its boundary B is held point-wise fixed. In cooperation with T. Minagawa, the speaker obtained theorems to the effect that the same rigidity phenomenon occurs for wide categories of non-convex shapes, and that in many instances it is sufficient to keep fixed only an appropriately chosen arc on the boundary.

3. *A step function solution of the bounded variation moment problem*, by Professor G. M. Bloom, Miami University.

G. Polya has shown that for arbitrary real sequences $\{\mu_n\}_0^\infty$ and $0=b_0<b_1<\cdots\rightarrow\infty$ there exist infinitely many step functions $\Phi(t)$ with points of change restricted to the set $\{b_n\}_0^\infty$ for which $\int_0^\infty t^n d\Phi(t)=\mu_n$, $\int_0^\infty |t^n d\Phi(t)|=B_n<\infty$, $n=0, 1, 2, \cdots$. Here for $\epsilon>0$, $a>1$, and $b_n=a^n$, $n\geqq 1$, is given a solution $\Phi(t)$ for which $B_0<|\mu_0|+\epsilon$.

4. *A probability distribution of random matches*, by Professor W. R. Van Voorhis, Fenn College.

If n numbered wires are attached at random one each to n correspondingly numbered terminals the probability that x of the numbers on the wires match the numbers on the terminals is $1/x!\sum_{r=0}^{n-x}(-1)^r/r!$, and the expectation of the number of matches is 1. This problem, expressed in other words, was first discussed by Montmort, and was later generalized by DeMoivre, Laplace, and Euler. Using a table for calculating the necessary moments, Professor Van Voorhis pointed out that the probability distribution has variance and skewness each equal to 1, as well as the mean, and that these statisties are independent of the integer n. Some properties of the distribution were discussed, and probability levels were given for exceeding certain numbers of matches.

5. *A new 23-place logarithm table to base 10 and e*, by Mr. E. R. Epperson, Miami University.

When accuracy to approximately twenty significant figures is desired in finding logarithms or antilogarithms, the factorizing method provides a procedure that decreases the arithmetical labor involved in such calculations. The new *Smithsonian Logarithmic Tables to Base e and Base 10*, Smithsonian Miscellaneous Collections, 118 (published by the Smithsonian Institution as publication 4054, 1952), prepared by G. W. Spenceley, R. M. Spenceley, and E. R. Epperson, is devoted exclusively to the factorizing method. This paper described the new volume containing the twenty-three decimal place values of the natural and common logarithms of integers from one to 10,000

and of appropriate decimal numbers and illustrated the application of the factorizing method to the volume.

6. *Social security and college retirement programs*, by Dr. O. L. Dustheimer, Cleveland, Ohio.

The author presented the results of a survey of pension plans now in effect in the colleges and universities of the United States and possessions. Of 815 institutions replying to the study questionnaire 81 per cent indicated that faculty members are now under social security.

7. *Some mathematical problems encountered in Air Force Research and Development work*, by Brigadier General L. I. Davis, Air Research and Development Command U.S.A.F. (By invitation).

Service research and development programs become involved in many problems requiring intensive mathematical treatment. The majority of the problems are common to the problems of science and engineering in any line of endeavor. A few examples of problems in mechanics requiring the use of differential equations, vector analysis, and probability relationships are given. The truly military problems involving tactics and strategy are held to be rather unsuitable for exact mathematical treatment. The many factors that enter into any attempt to make a model of strategic warfare are pointed out, using as an example a description of some of the components of an electronic war game built at the U.S.A.F. Institute of Technology, Wright Field, Dayton, Ohio.

8. *The law of inconsistency—a new approach to indirect proofs in mathematics*, by Professor Nathan Lazar, The Ohio State University.

The author discussed substitution for a proposition *A implies B* the related proposition *A and the denial of B are inconsistent*, and the application of this substitution to theorems in high school geometry where proofs are usually made by the indirect method.

9. *An interesting matrix*, by Mr. Fenton Stancliff, Akron, Ohio, introduced by the Secretary.

The author expressed e^x in a series of determinants, the n-th term of which was a determinant of the n-th order. When a positive integer K was added to each individual element of each determinant, the limit of the series reduced to $e^x/(K+1)$. This result was ascribed to a new invariant, tentatively referred to as the "lambda" number of a square array. When a single unit is added to each element of a determinant, this invariant fixes the amount of change in the determinant. Since the lambda number may be regulated at pleasure, the limit of any series of determinants may be regulated. The terms of the series for e^x were derived from "An interesting matrix," the subject of the talk, which was of such form, that all the characteristic roots and eigen-vectors were rational integral functions of the elements, and despite the possible high order of the matrix, were available by inspection.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The twelfth annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, on April 11 and 12, 1952.

Dr. Charles Wexler, Chairman of the Section, presided. Forty persons attended the sessions including the following twenty-six members of the Association:

J. W. Beach, Louis Brand, J. H. Butchart, B. R. Cato, Louise H. Chin, R. B. Crouch, G. A. Culpepper, W. W. Denton, D. G. Duncan, R. S. Fouch, R. F. Graesser, R. E. Graves, Max Kramer, H. B. Leonard, B. C. Meyer, W. W. Mitchell, J. L. Olpin, E. J. Purcell, Henry Schutzberger, Andrew Sobczyk, Deonisie Trifan, Earl Walden, D. L. Webb, R. L. Westhafer, Charles Wexler, Dale Woods.

At the business meeting the following officers were elected for the year 1952–1953: Chairman, J. H. Butchart, Arizona State College, Flagstaff; Vice-Chairman, M. S. Hendrickson, University of New Mexico. The Section accepted the invitation from the New Mexico College of A. and M. A. to hold the annual meeting at State College, New Mexico, January 2–3, 1953.

On Friday evening Dr. Louis Brand of the University of Cincinnati delivered an address entitled *The Inquiring Mind.*

The following papers were read in the two-day session:

1. *A theorem on double series*, by Professor B. C. Meyer, University of Arizona.

Let $\sum a_{ij}$ be a double series such that $|a_{ij}| \leqq |a_{mn}|$ for $i \geqq m, j \geqq n$ and such that each row and column is an alternating series. A necessary and sufficient condition that it be Sheffer convergent is that it be summable by diagonals. A comparison is made between Sheffer and regular convergence for series of this type.

2. *A minimum principle of plasticity*, by Professor Deonisie Trifan, University of Arizona.

For a given state of strain ϵ_{ij} throughout a body, and given velocities v_i on the surface such that the surface integral of the normal component vanishes, the rate of change of strain $\dot{\epsilon}_{ij}$, uniquely determined throughout the body, and $\dot{\sigma}_{ij}$ the corresponding rate of change of stress—$(\dot{\sigma}_{ij} - 1/3\delta_{ij}\dot{\sigma}_{kk} = 2G_0\dot{\epsilon}_{ij} - p(E)\dot{E}\epsilon_{ij}$ where $E = 2\epsilon_{ij}\epsilon_{ij}/3$, G_0 the shear modulus in the plastic range, and $p = p(E)$ an arbitrary function depending on the material)—satisfy the following minimum principle $\int_v \dot{\epsilon}_{ij}\dot{\sigma}_{ij}dv \leqq \int_v \dot{\epsilon}'_{ij}\dot{\sigma}'_{ij}dv$, where $\dot{\epsilon}'_{ij}$ are suitably selected strain rates and $\dot{\sigma}'_{ij}$ the corresponding stress rates.

3. *Conjugate functions in Boolean algebra*, by Professor Louise H. Chin, University of Arizona.

Let A be the set of elements in a Boolean algebra, and consider functions whose domain is A. A function g is said to be a conjugate function of a function f if the equations $f(x) \cdot y = 0$ and $g(y) \cdot x = 0$ are equivalent for any elements x and y in A. This concept of conjugate functions was introduced by A. Tarski and many consequences result from this definition. For example, this notion is especially useful when applied in the discussion of relation algebras and cylindric algebras.

4. *Length of a vertical column to a parabolic bridge*, by Professor W. W. Mitchell, Jr., Phoenix College.

In the design of a steel bridge with a parabolic arch it is necessary to calculate the length of the supporting columns from the arch to the roadbed. For an arch whose center line is a true parabola and whose cross-section is uniform, the center line of the curved surface upon which the vertical supporting columns rest is a curve at a uniform perpendicular distance from the parabolic center line of the arch. The equation of this curve was found, also an approximation, and the error made by using the approximate equation.

5. *The pi theorem of dimensional analysis*, by Professor Louis Brand, University of Cincinnati.

A dimensionally homogeneous or *isobaric* function $f(x_1, \cdots, x_n)$, in which the n variables x_i have the $m \times n$ dimensional matrix (a_{ij}) with respect to m fundamental units U_j, satisfies a certain identity in $t_1, t_2, \cdots, t_m$ when the units are changed from U_j to $U_j/t_j (t_j > 0)$. Ordinary homogeneous functions may be regarded as isobaric functions which have the dimensional matrix $(1, 1, \cdots, 1)$ with respect to a single unit U_1; in this case the corresponding identity in t is $f(tx_1, \cdots, tx_n) = t^a f(x_1, \cdots, x_n)$ where a is the degree of homogeneity. Putting $t = 1/x_1$ shows that the *equation* $f(x_1, \cdots, x_n) = 0$ is equivalent to $f(1, x_2/x_1, \cdots, x_n/x_1) = 0$, in which the $n-1$ arguments are dimensionless. A natural extension of this reasoning yields a proof of the pi theorem: *If the function f in an equation $f(x_1, \cdots, x_n) = 0$ with n arguments is isobaric with respect to m fundamental units $U_1, \cdots, U_m$, and if the $m \times n$ dimensional matrix of the arguments is of rank r, then if $n > r$, the given equation is equivalent to another $F(\pi_1, \cdots, \pi_{n-r}) = 0$ in $n-r$ independent and dimensionless products π_i formed from the original arguments.* The proof is constructive; and a simple matrix equation gives all the dimensionless products $\pi_1, \cdots, \pi_{n-r}$.

6. *The congruence of bisecants of the normal rational ruled surface of order* 4 *in* [5], by Professor E. J. Purcell, University of Arizona.

7. *The analysis and algebra of linear transformations*, by Dr. Andrew Sobczyk, Boston University and Los Alamos Scientific Laboratory.

The author reviewed the salient facts of the theory of (1) linear transformations between infinitely dimensional linear spaces, and (2) algebras of such transformations, contained in many published contributions on Hilbert space, Banach spaces and algebras. He then indicated his recent solutions of some outstanding problems of the theory.

8. *Torque and thrust on bounding cylinders in a two-dimensional flow problem*, by Professor J. W. Beach, University of New Mexico.

By starting with the known solution of $\nabla^4 \Psi = 0$ in the region between infinitely long eccentric cylinders in slow steady motion, the torque and thrust on a unit height of these cylinders was found.

9. *A note on the divergence of a certain Dirichlet series*, by Professor Dale Woods, Texas Western College.

The author gave a simple proof of the divergence of the series $\sum 1/n^z$ for $z = a + ib$ and $a \leqq 1$.

10. *Complex treatment of trochoidal curves*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

The author presented trochoids in general form with arbitrary radii of the fixed and rolling circles, using the *turn* of inversive geometry as the parameter. The self conjugate equation of the tangent was specialized for the deltoid and cardioid. The isotopic with a 60° angle was shown to be the trifolium and the orthoptic of the cardioid was shown to consist of a circle and a nodal limacon. The length of an arch and area under an arch of the general epicycloid were shown to be $8(a+b)b/a$ and $\pi b^2(3a+2b)/a$ respectively.

11. *Parabolic singular solutions of a differential equation of second degree*, by Professor R. L. Westhafer, New Mexico College of A. and M. A.

The p-discriminant locus for the differential equation $(a_0x + a_1y + a_2)p^2 + (b_0x + b_1y + b_2)p + (c_0x + c_1y + c_2) = 0$ is a second degree expression in x and y. The only cases in which the p-discrimi-

nant locus contains a non-degenerate conic for a singular solution of the differential equation are for certain parabolas.

12. *Statistical analysis of a department's grades*, by Professor R. F. Graesser, University of Arizona.

A record was kept of the distributions of semester grades in all classes in a department for a considerable period of time. These serve to establish a norm which may be adopted by the department as its ideal distribution. The combined grade distribution for all instructors in a given semester may be tested by the chi-square function for compatibility with this norm. If these distributions are incompatible, the instructors' individual distributions may be tested for homogeneity by the same chi-square function in order to determine whether this incompatibility is due to a change in the severity of grading by the whole department or by nonconformity of a few individuals. If the latter is suspected, then the distributions of individuals may be tested for compatibility with the norm. Changes in grading standards on the part of the whole department may be discovered by testing for trends with the theory of runs or with the serial correlation coefficient.

13. *Demonstration of space curves by means of light projection*, by Professors R. F. Graesser and E. J. Purcell, University of Arizona.

Space curves were beautifully demonstrated by use of planes and cones of light and string models of quadric, cubic, and quartic surfaces.

14. *Report of committee on high school cooperation*, by Professors Earl Walden, Max Kramer, New Mexico College of A. and M. A., and Charles Wexler, Arizona State College, Tempe.

Meetings have been held with the mathematics and science sections of the New Mexico Education Association and the Arizona Education Association. All agreed that better subject matter training was needed in the preparation of teachers. Plans have been set up for mathematical contests to be held in one of the New Mexico districts in 1953. If successful, expansion of these contests is anticipated.

15. *A subject matter course for the training of mathematics teachers*, by Professor Max Kramer, New Mexico College of A. and M. A.

A course was described, now being offered at New Mexico A. and M., in which subject matter on fundamentals from Courant and Robbins book *What Is Mathematics* is studied concurrently with the applications to the teaching of secondary school mathematics.

16. *Motor analysis*, by Professor Louis Brand, University of Cincinnati.

Dr. Brand gave an illuminating exposition of the theory of Motor Analysis and its application to the dynamics of a rigid body.

R. L. WESTHAFER, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

On April 25–26, 1952, the annual meeting of the Texas Section of the Mathematical Association of America was held in Commerce, Texas. East Texas State Teachers College served as sponsor.

About ninety persons attended the sessions of the Association including the following fifty-three members of the Association:

R. E. Baker, J. C. Bradford, Ina M. Bramblett, H. E. Bray, L. A. Colquitt, M. L. Coffman,

J. V. Cooke, A. H. Cowling, Don Cude, J. A. Daum, Jim Douglas, Jr., G. H. Dubay, Terrell Ellis, R. L. Glass, L. D. Gregory, W. T. Guy, Jr., E. H. Hanson, E. A. Hazlewood, E. R. Heineman, J. M. Hurt, E. C. Kennedy, H. A. Luther, E. K. McLachlan, W. K. McNabb, Hazel L. Mason, V. A. Miculka, Harlan C. Miller, E. D. Mouzon, Jr., M. E. Mullings, C. A. Murray, H. C. Parrish, R. B. Pinson, C. J. Pipes, Henry Rainbow, L. W. Ramsey, C. L. Riggs, R. Q. Seale, C. R. Sherer, F. W. Sparks, C. E. Springer, D. W. Starr, W. G. Stokes, E. J. Stulken, Jennie L. Tate, W. W. Taylor, M. E. Tittle, F. E. Ulrich, R. S. Underwood, Margaret M. Welch, Mabel Williams, H. A. Wood, A. W. Wortham, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor C. B. Wright, East Texas State Teachers College; Vice-Chairman, Professor L. A. Colquitt, Texas Christian University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

A banquet was given on Friday night; President J. G. Gee of East Texas State Teachers College was the principal speaker.

The program included the following papers:

1. *Some characterizations of certain set properties*, by Professor H. C. Parrish, North Texas State College.

For each of several (planar) set properties, a new characterization was given in terms of a (conditioned) region or sector property. Included were characterizations of a non-dense set; a non-dense, closed set; and a voidal set. A planar (linear) set S was termed voidal if for every point P of S there exists an open sector (interval) with P as vertex (end-point) and containing no point of S.

2. *The relaxation method as a projection method*, by Professor Samuel Agmon, The Rice Institute, introduced by the Secretary.

It is shown that the relaxation method used in the numerical solution of linear equations can be interpreted as a projection method. From this a simple geometric proof of the convergence of the relaxation method for a consistent system of linear equations is derived.

3. *On exhaustibility and certain related properties of a real function*, by Mr. J. C. Bradford, North Texas State College.

The speaker reported on a study made of certain function properties which were defined in terms of an exhaustible set, *i.e.* a set of the first category. In particular, functions which satisfy the property of Baire were studied. Most of the results were analogues of certain theorems on metric properties of functions.

4. *Stirling's approximation formula for* $n!$, by Professor C. L. Riggs, East Texas State Teachers College.

This is a development of Stirling's formula based upon the calculus of finite differences.

5. *An equivalence method for sequences of random variables*, by Mr. Jim Douglas, Jr., The Rice Institute.

A generalization of the classical truncation device due to Khintchine for the comparison of two sequences of independent random variables is introduced. The distance between two distribution functions $F(x)$ and $G(x)$ is defined to be the least upper bound of $|F(x)-G(x)|$ for all x. If $\{F_k(x)\}$ and $\{G_k(x)\}$ are, respectively, the sequences of distribution functions corresponding to two sequences of independent random variables, and if the sum of the distances between $F_k(x)$ and

$G_k(x)$ converges, the sequences are said to be equivalent. Applications to probability limit laws are discussed.

6. *Potential functions for particles moving with relativistic velocities in given orbits*, by Mr. M. L. Coffman, Texas Agricultural and Mechanical College.

The relativistic Lagrangian is used to determine the differential equation satisfied by the potential function required for particles moving in given orbits on given surfaces. The potential function satisfies a linear partial differential equation of second order. As examples, the potential functions are determined for two cases: where the curves are spirals on a cylinder, and where the curves are equilateral hyperbolas on a hyperbolic paraboloid.

7. *Union curves in differential geometry*, by Professor C. E. Springer, University of Oklahoma (by invitation).

A union curve on a surface relative to a given rectilinear congruence has the property that its osculating plane at every point contains the line of the congruence through the point. The union curvature vector of a curve on the surface is obtained. It is a zero vector in case the curve is a union curve, and it is the geodesic curvature vector in case the congruence is that of normals to the surface. A union affine connection appears, and its components reduce to the Christoffel symbols of the second kind for the congruence normal to the surface. The concepts of parallelism of surface vectors, intrinsic differentiation, and covariant derivative are generalized to union parallelism of surface vectors, union intrinsic differentiation, and union covariant derivative. An analogue of the Gauss-Bonnet theorem involving union curvature is introduced.

8. *Loci on the four-axes plane*, by Professor R. S. Underwood, Texas Technological College.

This paper discusses a coordinate system for four variables in which the positive halves of the z and u axes coincide respectively with the negative halves of the x and y axes. This scheme leads to very simple equations of transformation from the two-axes system. Also, since it provides the convenient lattice points of plane analytic geometry, it is well adapted to graphical discussions of simultaneous Diophantine equations in four variables. A method of finding typical, or area, loci is discussed, and a test for degenerate, or curve, loci is obtained.

9. *Differential equations of infinite order*, by Mr. T. M. Gallie, Jr., The Rice Institute, introduced by the Secretary.

This paper summarizes some interesting properties of solutions of homogeneous differential equations of infinite order with constant coefficients. These solutions are compared with those of the equation of finite order and a fundamental difference between the two is demonstrated. The existence of exponential solutions is shown and the possibility of expanding every solution throughout its domain of existence in a series of exponential solutions is discussed.

10. *A little-used method of solving a system of linear equations*, by Professor E. R. Heineman, Texas Technological College.

Let A be the matrix of a system of m linear equations in n unknowns. By performing elementary operations on the rows, but not the columns, of A (*i.e.* adding to any row a multiple of another, interchanging rows, multiplying a row by a constant), it is possible to obtain an equivalent matrix B, in which all elements below the principal diagonal of the coefficient matrix are zeros. The matrix B yields information as to the rank r of A, the consistency or inconsistency of the system, and the number of solutions. If the system is inconsistent, this completes the solution of the problem. For the consistent case, by manipulating the rows of B, it is possible to get an equivalent matrix C, in which all elements above the principal diagonal and appearing in the first r columns are zeros.

From matrix C, one can read the complete solution of the system. Slight variations of the method may be necessary in exceptional cases.

11. *Fractional integration and differentiation*, by Professor W. T. Guy, Jr., The University of Texas (by title).

Certain "fractional" operators were defined and applications of their use were given. The examples were chosen as being suitable for inclusion in the usual Laplace transform course for science majors.

12. *Computing the Kendall tau coefficient*, by Mr. H. F. Bright, The University of Texas (by title).

The assumptions implicit in the use of correlation techniques cannot always be justified, particularly for small samples. The Kendall tau coefficient offers a method of dealing with this difficulty since no assumptions regarding the shape of the distributions are necessary. However, the computation of this coefficient as ordinarily accomplished is laborious. In this note, a mechanical device is described for expediting the computation.

13. *The place of the mean value theorem in the differential calculus course*, by Mr. J. M. Hurt, The University of Texas.

The mean value theorem is fundamental in the study of increasing and decreasing functions, maxima and minima, and Newton's method, as well as in the development of L'Hospital's rule. Thus its early introduction into the differential calculus course strengthens the treatment of these important topics, gives the student a powerful tool with which to make his arguments more rigorous and complete, and obviates the temptation to give discussions that are vaguely intuitive, incomplete, or incorrect.

14. *A program of correlated mathematics in secondary schools*, by Sister Antonietta, Incarnate Word College, introduced by the Secretary.

By its nature mathematics is cumulative. What is studied in one branch should help towards understanding the next. While arithmetic, algebra, geometry, trigonometry are distinct to the extent that all subject matter boundaries cannot be dissolved, yet the connections between them not only unify the whole field of mathematics but also give a better grasp of each phase. To make these connections where they occur naturally is the aim of correlated mathematics. This program is worked out on the basis of the recommendations of the 1940 Reorganizations Committee which was appointed by the National Council of Mathematics Teachers.

15. *What do mathematicians do?* by Professor W. T. Guy, Jr., The University of Texas.

Since no adequate definition of mathematics has been given (nor is likely to be given) the author made an attempt to delineate what mathematicians do. The role of the pure and the applied mathematician was examined. It was suggested that such an orientation during the usual college course would help the non-mathematics majors (as well as the mathematics majors) in understanding and appreciating the place of mathematics in their own fields.

16. Panel discussion: *Certification of teachers in Texas*, by Professor J. E. Franklin and Professor Everett Shepherd, East Texas State Teachers College; Professor H. E. Bray, The Rice Institute; Professor C. R. Sherer, Texas Christian University; Professor E. A. Hazelwood, Texas Technological College, Moderator.

C. R. SHERER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, College of St. Scholastica, Duluth, October 11, 1952.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City, October 31, 1952.

PACIFIC NORTHWEST

PHILADELPHIA, University of Delaware, Newark, November 29, 1952.

ROCKY MOUNTAIN

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

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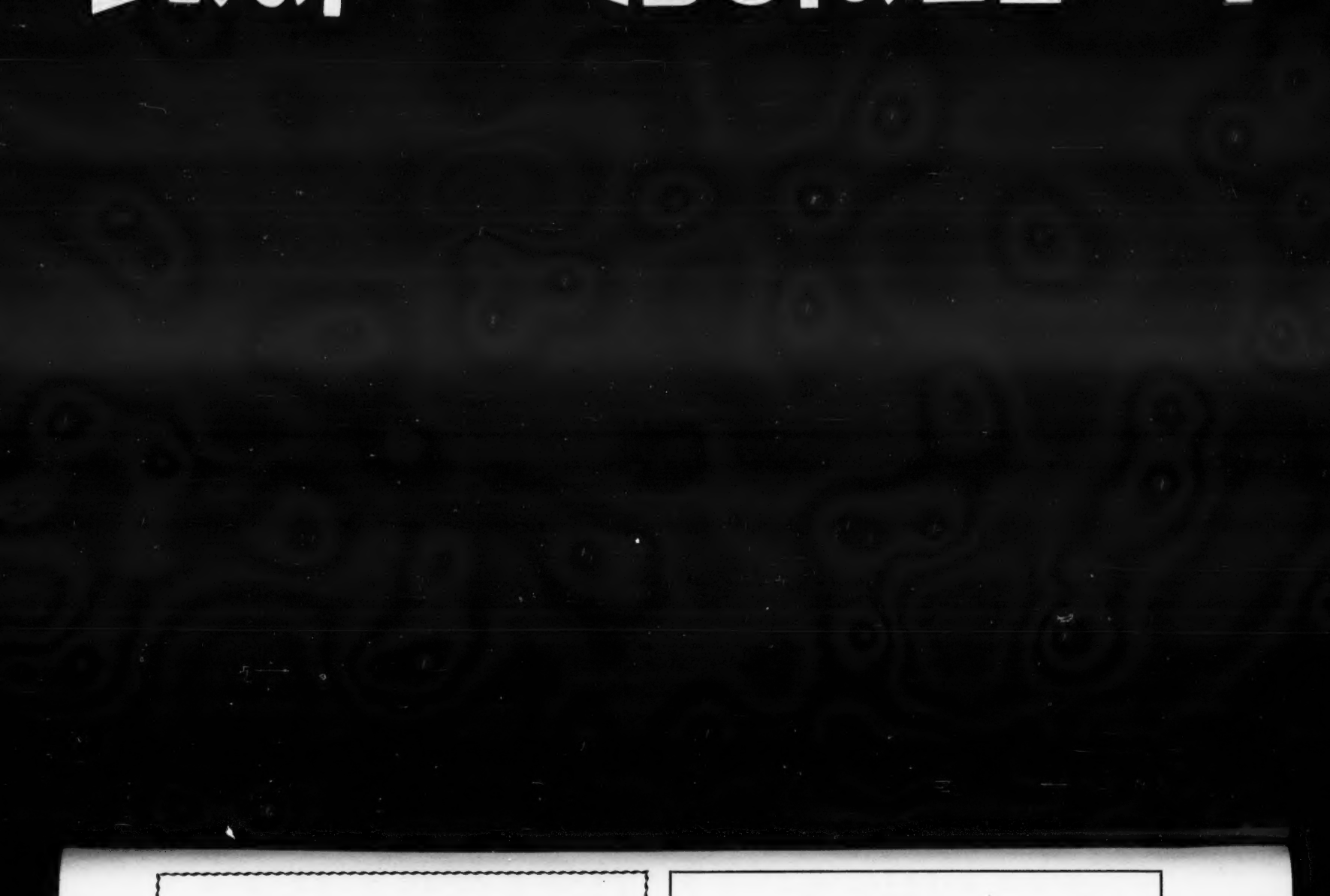

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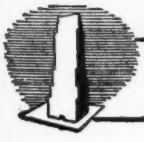

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